HILGARDIA

A JOURNAL OF AGRICULTURAL SCIENCE PUBLISHED BY THE CALIFORNIA AGRICULTURAL EXPERIMENT STATION

Volume 47, Number 4 *September, 1979*

Scaling of Field-measured Soil-water Properties

I. Methodology
II. Hydraulic Conductivity and Flux

C.S. Simmons, D.R. Nielsen, and J.W. Biggar

UNIVERSITY OF CALIFORNIA DIVISION OF AGRICULTURAL SCIENCES

Concepts of similar soil and scaling are applied to investigate the spatial variability of the field-measured soil-water properties, soil-water pressure head, hydraulic conductivity, and soil-water diffusivity associated with unsaturated flow. The classical, analytical aspects of scale factors as regards the invariance of the flow equations expressed in terms of "reduced variables" are reviewed and extended by considering stochastic aspects of random variations in soil-water properties. It is demonstrated that scaling can best be achieved when soil-water properties are represented by a set of related model functions.

The scale distribution is obtained from soil-water pressure head and water content measurements for soil sampling locations 30, 60, 120, 180, 240, and 300 cm below the surface in 12 plots planted to corn. Scale factors are found to have an approximate log-normal distribution.

Methods of computing scale factors directly from soil-water pressure head and hydraulic conductivity measurements and indirectly from soil-water content profiles for a drainage experiment are derived. Improved methods for estimating unsaturated hydraulic conductivity are also presented. Stochastic behavior of flux and cumulative seepage as random functions of the scale factors for a similar soil is described for a simple drainage experiment.

THE AUTHORS:

C.S. Simmons, formerly Post Graduate Research Water Scientist, Department of Land, Air, and Water Resources, Davis, is Research Scientist, Geosciences Department, Battelle Northwest Labs, Richland, Washington.

D.R. Nielsen is Professor of Water Science, Department of Land, Air, and Water Resources, Davis.

J.W. Biggar is Professor of Water Science, Department of Land, Air, and Water Resources, Davis.

TABLE OF CONTENTS

II. Hydraulic Conductivity and Flux

In contrast to the soil-water pressure head, which is a directly measurable and fundamental soil-water property, the hydraulic conductivity must first be computed from soil-water flow measurements before scaling of conductivity can be verified. Moreover, the estimated conductivity can vary drastically depending on the method of calculation employed. Such computational variation can contribute to spatial variation and can alter the results of scaling. For this reason a number of methods for computing conductivity are compared. These methods are based on ability to predict flux consistently from available estimates of the pressure head gradient.

Scaling of conductivity is verified here for the measurements of the 1-m plots drainage experiment. A previous experiment of similar design (Nielsen, Biggar and Erh) 1973) has indicated that measurement of pressure head at only two depths in the profile of each plot is sufficient to estimate conductivity at intermediate depths. Measurement uncertainty allows for incomplete evaluation of pressure head gradient over a profile. Therefore, the pressure head for the 1-m plots was measured at only 60 and 120 cm.

Scaling of flux is demonstrated with modifications in the scale relations of Miller and Miller (1956). Flux is scaled by scaling the measured water content profiles of the 1-m plots. Stochastic behavior of flux as a random function of an exponential conductivity model has been described by Warrick, Mullen, and Nielsen, (1977*b*). Their approach uses a simplified drainage equation and requires averaging over all parameters. Here it is demonstrated that description of stochastic behavior is considerably simplified by representing spatial variability by scale factors. The method applies to a field of similar soil, as found in the experimental field.

Calculation of Conductivity: Theoretical

Direct method

Conductivity can be computed directly by the instantaneous profile method using flow data from the drainage experiment. The general method is similar to that described in detail by Nielsen, Biggar and Erh (1973).

The equation for drainage and redistribution in a covered soil profile is

$$z \frac{\partial \hat{\theta}}{\partial t} = K(\theta, z) \left[1 + \frac{\partial h}{\partial z}\right] \tag{1}$$

where

$$\hat{\theta} = \frac{1}{z} \int_0^z \theta(z,t)\, dz \tag{2}$$

defines the depth-averaged water content and z is the depth below the soil surface. Equation (1) is an integral of Richards' equation with the boundary condition that surface flux equals zero. The flux is

$$J = z \frac{\partial \hat{\theta}}{\partial t} \tag{3}$$

and according to equation (1) the conductivity $K(\theta,z)$equals the flux divided by the hydraulic gradient.

Finite difference estimates of $K(\Theta,z)$

Suppose that the water contents θ and pressure heads h are measured at n times t_1, t_2, ..., t_n and a k depths (nodes z_1, z_2, ..., z_k below the surface, denoted by $z_o = 0$. Then, for each depth $z = z_j$ (j = 1, ..., k), the conductivity $K(\theta_i,z)$ for water content $\theta(z,t_i)$ (i = 1, ..., n) is estimated from

$$z\left(\frac{\hat{\theta}_{i+1} - \hat{\theta}_i}{t_{i+1} - t_i}\right) = K(\theta_i, z)\left[1 + \frac{\partial h}{\partial z}(z,t_i)\right] \tag{4}$$

where $\hat{\theta}_i = \hat{\theta}(z,t_i)$ and

$$\frac{\partial h}{\partial z}(z,t) \doteq \frac{1}{2}\left(\frac{\Delta h_j}{\Delta z_j} + \frac{\Delta h_{j+1}}{\Delta z_{j+1}}\right)$$

with

$$\Delta h_j = h(z_j,t) - h(z_{j-1},t)$$

and

$$\Delta z_j = z_j - z_{j-1}.$$

Equation (4) gives an advanced time, t_{i+1}, estimate of water content based on a present time, t_i, estimate of conductivity and head gradient. A time averaged estimate of conductivity, $K(\bar{\theta}_i, z)$, can be computed with the averages

$$\bar{\theta}_i = \frac{1}{2}(\theta_{i+1} + \theta_i)$$

and

$$\overline{\frac{\partial h}{\partial z}}(z,t_i) = \frac{1}{2}\left(\frac{\partial h}{\partial z}(z,t_i) + \frac{\partial h}{\partial z}(z,t_{i+1})\right)$$

used in equation (4).

In the case that pressure head is measured for only two depths z′ and z″, the conductivity at an intermediate depth z can be estimated from

$$z\left(\frac{\hat{\theta}_{i+1} - \hat{\theta}_i}{t_{i+1} - t_i}\right) = K(\theta_i, z)\left[1 + \frac{\Delta h_i}{\Delta z}\right] \tag{5}$$

where

$$\frac{\Delta h_i}{\Delta z} = \frac{h(z'',t_i) - h(z',t_i)}{z'' - z'} \tag{6}$$

and

$$z\hat{\theta}(z,t) = \sum_{j=1}^{J} \frac{1}{2}(\theta(z_{j-1},t) + \theta(z_j,t))\,\Delta z_j \tag{7}$$

for $z = z_j$, j = 1, ..., k. Equation (7) is an estimate of cumulative storage to depth z.

At the last depth z″, the pressure head gradient is estimated as half of equation (6), so that the pressure head gradient is assumed zero below z″.

A difficulty with the direct finite difference methods is that measurement uncertainty can produce extreme variability in conductivity estimates and even nonsense estimates, such as negative conductivity.

Analytical estimates of conductivity

When the condition of the data does not allow direct finite difference estimation of conductivity, the pressure head gradient and flux can be smoothed and conductivity computed analytically with the following methods.

Assuming scaled soil-water characteristics, the pressure head gradient can be estimated from the scales α and the measured water-content profile with the following:

$$\frac{\partial h}{\partial z} = -\frac{h_m(\theta)}{\alpha^2}\frac{d\alpha}{dz} + \frac{1}{\alpha}\frac{\partial h_m(\theta)}{\partial \theta}\frac{\partial \theta}{\partial z} \tag{8}$$

where h_m is the scale mean pressure head, which is explicitly independent of z. If the rate of change of α with depth is negligible, then equation (8) becomes approximately

$$\frac{\partial h}{\partial z} = \frac{1}{\alpha}\frac{\partial h_m(\theta)}{\partial \theta}\frac{\partial \theta}{\partial z}. \tag{9}$$

An average pressure head gradient is obtained by averaging (9) over the scales.

If the soil-water characteristics are measured at only two depths, z′ and z″, an estimate of pressure head gradient is given by

$$\frac{\partial h}{\partial z} \doteq \frac{h_m(\theta'')/\alpha'' - h_m(\theta')/\alpha'}{z'' - z'} \tag{10}$$

where α', and $\alpha''$are the scales and θ', and θ'' the water contents for those depths. Equation (10) is the estimate applicable to the 1-m plots. The pressure head gradient (10) is independent of depth, but depends on time through θ' and θ''.

The water content θ and its time dependence can be expressed as a polynomial regression for each location and depth as follows:

$$\theta = \sum_{k=0}^{p} a_k t^k \tag{11}$$

where the regression coefficients a_o, a_1, ..., a_p depend on the location and depth. Estimates of the water content gradient and depth-averaged water content are then given by

$$\frac{\partial \theta}{\partial z} = \sum_{k=0}^{p} \frac{\partial a_k}{\partial z} t^k \tag{12}$$

and

$$\hat{\theta} = \sum_{k=0}^{p} \hat{a}_k t^k \tag{13}$$

where

$$\hat{a}_k = \frac{1}{z}\int_0^z a_k \, dz \qquad (k = 0, 1, \ldots, p).$$

Usual finite difference methods are applied to compute derivatives and integrals of the regression coefficients over the nodes of the measured water content profile. An estimate of flux is then given by

$$J(z,t) = z \sum_{k=1}^{p} k \hat{a}_k t^{k-1}. \tag{14}$$

The conductivity is obtained from the drainage equation (1) by using the estimate of hydraulic gradient (9) in conjunction with the regression estimates (11) through (14). Then

$$K(\theta,z) = J(z,t)\left[1 + \frac{1}{\alpha}\frac{\partial h_m(\theta)}{\partial \theta}\frac{\partial \theta}{\partial z}\right]^{-1} \tag{15}$$

gives the conductivity as an explicit function of time t (and an implicit function of θ) for each depth z. The conductivity can be found as an explicit function of water content by using the regression

$$t = \sum_{k=0}^{q} b_k \theta^k \tag{16}$$

in addition to (14). Flux as an explicit function of θ is

$$J(\theta,z) = z \sum_{k=1}^{p} k \hat{a}_k \left[\sum_{j=0}^{q} b_j \theta^j\right]^{k-1}, \tag{17}$$

and the water content gradient is given by

$$0 = \sum_{k=0}^{q} \frac{\partial b_k}{\partial z}\theta^k + \left(\sum_{k=1}^{q} k b_k \theta^{k-1}\right)\frac{\partial \theta}{\partial z} \tag{18}$$

or by substitution of (16) into (12). Substitution of (17) and (18) into (15) gives the conductivity as an explicit function of θ.

Other model functions for describing the time dependence of water content can be applied in a similar way in place of (11). However, an advantage of (11) is the linear dependence on regression coefficients. Standard regression methods are used to fit (11), and the covariance matrix of the regression coefficients can be used to analyze propogation of measurement errors. A disadvantage of the polynomial regression (11) is the dependence of the derivatives (12) and (14) on the particular choice of degree p. A p too large will result in over-fitting of the data and a p too small will result in an incorrect time dependence model. For the 1-m plots data, it was found that a simple nonlinear model used by Nielsen, Biggar, and Erh (1973) best described the water content. Application of this model is discussed in the next sections.

Regression of depth-averaged water content

It has been suggested by Nielsen, Biggar, and Erh (1973) that depth-averaged water content (2) approximately equals water content at each depth. Now consider the situation, during a drainage experiment, for which this suggestion is valid. Thus, suppose that the drainage water content profile undergoes a parallel translation in time:

$$\theta(z,t) = \theta_o(z) + \delta(t) \tag{19}$$

where $\theta_o(z)$ is an initial profile, and δ depends only on time. Then integration of (19) gives

$$\hat{\theta}(z,t) = \hat{\theta}_o(z) + \delta(t) \tag{20}$$

and

$$\hat{\theta}(z,t) = \theta(z,t) + d \tag{21}$$

where $d = \hat{\theta}_o - \theta_o$. If the initial depth-averaged water content equals the water content, i.e., $d = 0$, then $\hat{\theta} = \theta$ during the entire drainage at each depth.

So as to account for water content profiles that translate in time without remaining parallel, a correction coefficient c is introduced into (21) to give

$$\hat{\theta}(z,t) = c\theta(z,t) + d. \tag{22}$$

Indeed, it is found that the drainage profile satisfies a linear regression given by (22) where c and d depend only on depth z. For the 1-m plots, equation (22) holds with regression coefficients c in the range 0.6 to 1.0 and with correlations about 0.95, and estimates $\hat{\theta}$ from θ within measurement error.

By applying the regression (22), the flux J(z,t) is related to the rate of change in water content according to the relation

$$J(z,t) = zc \frac{\partial\theta}{\partial t} \tag{23}$$

Thus, drainage and redistribution take place according to the following stochastic first-order partial differential equation:

$$zc \frac{\partial\theta}{\partial t} = K(\theta,z) \left(1 + \frac{\partial h}{\partial z}\right). \tag{24}$$

Unit hydraulic gradient drainage equation

Often a valid assumption for drainage of a profile at deep depths is that the hydraulic gradient is unity, or the pressure head gradient is zero. Assuming an exponential conductivity model and unity hydraulic gradient, the drainage equation (24) becomes

$$zc \frac{\partial\theta}{\partial t} = K_o e^{\beta(\theta-\theta_o)}. \tag{25}$$

where K_o and θ_o are the steady state infiltration values, or initial values. Integration of equation (25) with z positive below the soil surface gives

$$\theta = \theta_o - \frac{1}{\beta} \ln[1 + \beta K_o t/cz] \tag{26}$$

and

$$\hat{\theta} = \hat{\theta}_o - \frac{1}{\hat{\beta}} \ln[1 + \hat{\beta} K_o t/z] \tag{27}$$

where $\hat{\beta} = \beta/c$.

Equations (26) and (27) were found to fit the water-content profile at every depth in the 1-m plots. This result indicates that (26) and (27) apply with generality that goes beyond the original assumptions, because hydraulic gradient is not unity over the entire profile. In view of the flux equation (23), the implication is that flux is an exponential functional of the form

$$J(\theta,z) = J_o e^{\delta(\theta-\theta_o)} \tag{28}$$

where J_o is the steady state flux or initial flux, and the least squares fit parameters K_o and β are estimates of J_o and δ, respectively. Thus, the parameters J_o and δ represent the exponential conductivity model only when the pressure gradient is zero. The general drainage equation applicable to the entire profile is then

$$\theta = \theta_o - \frac{1}{\delta} \ln[1 + J_o \delta t/cz] \tag{29}$$

and flux as an explicit function of time is given by

$$J(z,t) = \frac{J_o}{1 + \delta J_o t/cz}. \tag{30}$$

Least squares fit of the drainage equation (29) provides an indirect verification of the model (28). Although not given here, we have verified (28) directly by a least squares fit to finite difference estimates of the flux.

The representations of the flux, equations (28) and (30), can be substituted into equation (1) along with any of the given estimates of pressure head gradient to yield the conductivity. A drainage equation (29) was found to be superior to the polynomial regression (11), at least for the 1-m plots.

Scaling of Conductivity

Conductivity models

Conductivity is often represented within experimental measurement error for a restricted range of water content by the exponential function

$$K = K_o e^{\beta(\theta-\theta_o)} \tag{31}$$

where θ_o is some reference value of water content, and K_o is the corresponding conductivity. This result is common in the literature. In each case, the model is verified by least squares fit of its linear form:

$$\ln K = \ln K_o + \beta(\theta-\theta_o). \tag{32}$$

For the 1-m plots, the direct finite difference estimates of conductivity based on the pressure head gradient between two depths satisfy a regression (32) for each location.

In terms of the degree of saturation s, the conductivity function is

$$K = K_o \, e^{b(s-s_o)} \tag{33}$$

where K_o is now the conductivity at the reference saturation s_o. Another representation of conductivity is obtained by defining s in (33) as an approximate saturation equal to θ/θ_o, where θ_o is the water content during steady state infiltration. With s_o equal to 1, the models (31) and (33) are then related by $\beta = b/\theta_o$. These models yield equivalent descriptions of conductivity when applied independently to each location with given values of θ_o and saturated water content ϕ.

The conductivity can be scaled for a set of locations if either of the exponential models can be least squares fit with a common value of β or b for all locations. As a consequence of measurement uncertainty, the conductivity can scale possibly with respect to both models, even though the scaling results are not equivalent. In general, the scaling with respect to these models is not equivalent, because each emphasizes a different parameter as being common for all locations, and the scale factors obtained with each model need not be equal. Scaling will yield equivalent results for the two models only if θ_o and ϕ are constant over all locations.

Conductivity scale relation

Scale factors ω for the conductivity models (31) and (33) are defined relative to common reference values of θ_o and s_o. To obtain the scales, the models (31) and (33) can be expressed in terms of averages $\bar{\theta}_o$ and $\bar{s}_o$ over locations as follows:

$$K = \bar{K}_o \, e^{\beta(\theta-\bar{\theta}_o)} \tag{34}$$

where

$$\bar{K}_o = K_o \, e^{\beta(\bar{\theta}_o-\theta_o)} \tag{35}$$

and

$$K = \bar{K}_o \, e^{b(s-\bar{s}_o)} \tag{36}$$

where

$$\bar{K}_o = K_o \, e^{b(\bar{s}_o-s_o)}. \tag{37}$$

The transformed models (34) and (36) now have the same reference value of θ and s. Assuming that the measured conductivity for a set of locations is scaled with either of the exponential models, the scale relation is

$$\bar{K}_o = \omega^2 K_m \tag{38}$$

where ω is the scale factor of each location and K_m is the scale mean value of all $\bar{K}_o$. For a set of R locations, the scale mean K_m is determined by the condition that the average of ω over locations equals unity. The scale mean is then given by

$$K_m^{1/2} = \frac{1}{R} \sum_{r=1}^{R} \bar{K}_{or}^{1/2} \tag{39}$$

where subscript r denotes the location. Values of the scales ω_r (r = 1, ..., R) are then established by the scale relation (38), for each location.

Scaling of conductivity with respect to the exponential models (34) and (36) has an important property: values of ω do not depend on the particular common values of $\bar{\theta}_o$ and $\bar{s}_o$ used. That is, scaling results are invariant with respect to the choice of $\bar{\theta}_o$ and $\bar{s}_o$. However, values of K_o do depend through (35) and (37) on the particular θ_o and s_o used. Moreover, the domains of the models are still determined by θ_o and s_o of each location.

A different symbol ω is used to distinguish conductivity scales and pressure head scales α. Indeed, the principle of scaling as proposed by Miller and Miller (1956) is valid if, within limits of statistical and computational error, the conductivity scales ω equal the pressure head scales α for every location. Equality of these scales is not a direct consequence of Richards' equation, but further depends on the scaling behavior of flux.

Scaling in terms of water content

Measured conductivity of the 1-m plots was scaled with the model (33) for an approximate saturation variable s equal θ/θ_o and for s_o equal 1. One would expect that scales computed in terms of s will approximate those in terms of θ if θ_o does not vary much over locations. This possibility will now be considered.

The following method provides an estimate of the effect on scales of changing conductivity models. Suppose that conductivity is scaled for R locations with

$$K_r = K_{or}\, e^{b(s-1)} \qquad (r = 1, \ldots, R) \tag{40}$$

where $s = \theta/\theta_{or}$, and b is common for all locations. A common θ_o is defined by

$$\frac{1}{\theta_o} = \frac{1}{R}\sum_{r=1}^{R}\frac{1}{\theta_{or}}. \tag{41}$$

Let $\beta_r = b/\theta_{or}$. Then in terms of θ, (40) becomes

$$K_r(\theta) = K_{or}\, e^{\beta_r(\theta-\theta_{or})} = \hat{K}_{or}\, e^{\beta_r(\theta-\theta_o)} \tag{42}$$

where

$$\hat{K}_{or} = K_o\, e^{\beta_r(\theta_o-\theta_{or})}. \tag{43}$$

Now define

$$\hat{K}_r(\theta) = \hat{K}_{or}\, e^{\beta(\theta-\theta_o)} \qquad (r = 1, \ldots, R) \tag{44}$$

with

$$\beta = \frac{1}{R}\sum_{r=1}^{R}\beta_r = b/\theta_o. \tag{45}$$

Then (44), which is scaled in terms of θ, is an approximation to (40). New scales $\hat{\omega}$ are determined by

$$\hat{K}_{or} = \hat{\omega}_r^2\,\hat{K}_m \tag{46}$$

with

$$\hat{K}_m^{1/2} = \frac{1}{R} \sum_{r=1}^{R} \hat{K}_{or}^{1/2} \tag{47}$$

The standard deviation of β_r denoted σ_β, which equals b multiplied by standard deviation of $1/\theta_{or}$, represents error in β. Scales $\hat{\omega}$ are related to the original scales ω by

$$\hat{\omega}_r^2 = \omega_r^2 \, (K_m/\hat{K}_m) \, e^{b(\theta_o/\theta_{or}-1)}. \tag{48}$$

The difference in (42) and (44) for each location is given by

$$\ln \hat{K}_r(\theta) - \ln K_r(\theta) = (\beta-\beta_r)(\theta-\theta_o). \tag{49}$$

Thus the standard deviation over locations of the logarithm of conductivity equals

$$\sigma_\beta \, |\theta-\theta_o|. \tag{50}$$

Equation (50) constitutes an estimated deviation of the models when scaling representations are changed from saturation to water content.

Of course, scales in terms of θ can be obtained directly by least squares fit of (31) with a common β. Scales $\hat{\omega}$ are not expected to remain valid approximations of the directly obtained scales when θ_{or} have large variation.

Indirect scaling of flux

The scaling of conductivity implies certain indirect scaling results for the water content drainage profile and the flux. Consider a drainage profile with unit hydraulic gradient, and suppose that θ_o is constant throughout the profile, so that the scale relation becomes

$$K_o = \omega^2 \, K_m. \tag{51}$$

Let z_m denote a reference depth, for example, the last depth in the profile. A macroscopic length scale L is then defined by $L = z/z_m$. A reduced time τ for each location will be defined by

$$\tau = (\omega^2/cL)t. \tag{52}$$

Then, in terms of reduced time, the water content profile (26) is described by the same equation:

$$\theta = \theta_o - \frac{1}{\beta} \ln[1 + \beta K_m \tau/z_m]. \tag{53}$$

At each location, the flux is given by

$$J(z,t) = \omega^2 \, J_m(\tau) \tag{54}$$

where

$$J_m(\tau) = \frac{K_m}{1 + \beta K_m \tau/z_m} \tag{55}$$

defines the reduced flux.

Equation (54) constitutes a scale relation for flux (23), and provides an indirect method for scaling conductivity. Conductivity based on unit hydraulic gradient can be scaled

with an exponential model if (26) can be least squares fit to measured water content of all locations with a common β. The fit estimates K_o for each location, and the scales are obtained from (38), using (35). An assumption of constant θ_o is not required here.

In general, however, the hydraulic gradient is not unity throughout the entire profile, so that the fit of (26) actually represents equation (29) with a common δ. Thus the flux (28) can be scaled if the drainage equation (29) can be least squares fit with a common δ for all locations. Then drainage scales ω are defined by

$$\bar{J}_o = \omega^2 J_m \tag{56}$$

where

$$\bar{J}_o = J_o \, e^{\delta(\bar{\theta}_o - \theta_o)} \tag{57}$$

for each location, and $\bar{\theta}_o$ is an average reference steady state θ_o. The scale mean flux J_m is determined relative to the common $\bar{\theta}_o$:

$$J_m^{1/2} = \text{average } \bar{J}_0^{1/2}. \tag{58}$$

Equation (58) follows from the constraint that average ω equals one. Reduced time is then defined as

$$\tau = \left(\frac{\omega^2}{cL} \, e^{\delta(\theta_o - \bar{\theta}_o)} \right) t. \tag{59}$$

In terms of reduced time, the water content profile is described over all locations by

$$\theta = \bar{\theta}_o - \frac{1}{\delta} \ln[1 + \delta J_m \tau / z_m]. \tag{60}$$

Hence, the water content profiles for a set of similar soil locations can be coalesced into a single curve (60) by scaling time according to (59) and by transforming the water-content reference from θ_o to $\bar{\theta}_o$. Dependence of flux on reduced time is given by

$$J(z,t) = \frac{\omega^2 J_m \, e^{\delta(\theta_o - \bar{\theta}_o)}}{1 + \delta J_m \tau / z_m} \tag{61}$$

New scale relations

Scaling of time and flux as in equations (52) and (54) suggests that another set of reduced variables, which differ from those of Miller and Miller (1956), can be defined. The flow equations are written as

$$\frac{J}{\alpha^2} = - \frac{K}{\alpha^2} \frac{\partial}{\partial z} [z + h] \tag{62}$$

and

$$L \frac{\partial}{\partial z} (J/\alpha^2) = - \partial\theta / \partial(\alpha^2 t / L). \tag{63}$$

Then invariance of Darcy's law (62) and the continuity equation (63) implies the following scale relations:

$$\text{(i)}\quad h_m = \alpha h$$

$$\text{(ii)}\quad K_m = K/\alpha^2$$

$$\text{(iii)}\quad z_m = z/L$$

$$\text{(iv)}\quad J_m = J/\alpha^2$$

$$\text{(v)}\quad t_m = \alpha^2 t/L$$

The subscript m denotes a reduced variable, i.e., a scale mean quantity, and α is the scale factor. These scale relations differ only in the definitions of reduced flux (iv) and reduced time (v). In terms of reduced variables, the flow equations are

$$J_m = - K_m \left[1 + \frac{1}{\alpha L} \frac{\partial h_m}{\partial z_m} \right] \tag{64}$$

and

$$\frac{\partial J_m}{\partial z_m} = - \frac{\partial \theta}{\partial t_m}. \tag{65}$$

Thus, the solution of the flow equations is invariant over locations if the pressure head gradient is invariant; this is the case, for example, if $\partial h/\partial z = 0$. Note that the invariance of (62) and (63) assumes a scale-homogeneous soil profile, or that the gradient of α is negligible. In general, the reduced flux (64) is invariant over locations if αL is invariant, and this is the same condition required by the original Miller and Miller definition.

Indirect estimate of conductivity and scales based on pressure head for two depths

The following method for scaling conductivity avoids the difficulty of evaluating the hydraulic gradient over a profile.

Soil-water characteristic curves h′ and h″ are assumed determined at two depths z′ and z″. Then the conductivity at profile depths between z′ and z″ for which drainage as a function of time is measured can be calculated as follows. An integral equation for $K(\theta, z)$ is given by

$$\int_{z'}^{z''} \frac{J(\theta,z)}{K(\theta,z)}\, dz = z'' - z' + h'' - h'. \tag{66}$$

The water content θ is provided as data for the n depths $z' = z_1, z_2, \ldots, z_{n-1}, z_n = z''$, and the flux $J(\theta,z)$ is described as a functional of $\theta(z,t)$, or determined as an explicit function of time, $J(z,t)$, at each depth. For example, a polynomial representation of flux such as (14) can be used. Now let

$$f(z) = J(\theta(z,t),z)/K(\theta(z,t),z) \tag{67}$$

where conductivity $K(\theta,z)$ is described by some choice of model function. A finite difference approximation of the integral (66) over the profile is given by

$$\int_{z'}^{z''} f(z)dz = f(z_1) \frac{\Delta z_1}{2} + \sum_{j=2}^{n-1} f(z_j) \frac{(\Delta z_j + \Delta z_{j-1})}{2} + f(z_n) \frac{\Delta z_{n-1}}{2}. \tag{68}$$

Each $f(z_j)$ ($j = 1, \ldots, n$) is an explicit function of time t, which depends on the parameters $b_1, \ldots, b_p$ of the conductivity model $K(\theta, z; b_1, \ldots, b_p)$ for each location. Thus equation (66) with the approximation (68) depends on the np parameters $b_{11}, \ldots, b_{p1}$; $b_{12}, \ldots, b_{p2}$; $\ldots$; $b_{1n}, \ldots, b_{pn}$. These unknown parameters can be estimated by a direct nonlinear least squares fit of (66) using (68). The integral (68) is fit to the difference in hydraulic potential, $z + h$, as a function of time.

In particular, the model (31) depends on two parameters: K_o and β. With the flux model (28), (67) becomes

$$f(z) = \frac{J_o}{K_o} e^{(\delta-\beta)(\theta-\theta_o)}. \tag{69}$$

where all parameters may depend on z. Then conductivity can be estimated and scaled simultaneously by letting

$$b_j = 1/\omega_j^2 K_m \qquad (j = 1, \ldots, n)$$

and by taking $b_{n+1} = \beta$ as the common parameter. Then

$$f(z_j) = b_j J_{oj} e^{(\delta_j-\beta)(\theta_j-\theta_o)}. \tag{70}$$

for $j = 1, \ldots, n$ where $\theta_j = \theta(z_j,t)$, J_{oj}, and δ_j are known from measurement.

Since the drainage flux can be scaled for the 1-m plots, δ_j corresponding to each location can be replaced by a single common δ. By the scale relations, a single average value of θ_o can be used.

Disadvantages of this method are that a prior choice of conductivity model causes biased estimates and that a requirement of many parameters $b_1, \ldots, b_n$ can make a unique and accurate fit impossible.

Corrected Estimates of Conductivity

In circumstances for which direct measurements of the entire pressure head profile are not available, the following methods are used to improve estimates of conductivity by estimating hydraulic gradient from the scaled water-content profile.

Conductivity based on estimate of pressure head gradient

Usually the soil profile of a field is scale heterogeneous, and computation of the pressure head gradient requires evaluation of the scale factor's derivative. However, when the pressure head scales, α, are incomplete over the profile, as is the case for the 1-m plots, the derivative $d\alpha/dz$ cannot be estimated properly. But assuming that the new scale relations are valid, the scales α are identified with those obtained by scaling drainage according to (56). Then assuming that α equals ω for all locations, the gradient of α can be estimated over the same profile for which θ is measured.

Suppose that the parameters of the pressure head model are estimated for at least one location and that the drainage profile is scaled with (56) and (60). Then the head gradient is

$$\frac{\partial h}{\partial z} = -\frac{h_m}{\alpha}\left(\frac{1}{\alpha}\frac{d\alpha}{dz}\right) + \frac{a_m}{\alpha}\rho e^{\rho(\theta-\phi)}\frac{\partial\theta}{\partial z} \tag{71}$$

where the pressure head model is

$$h_m = a_m \left(e^{\rho(\theta-\phi)} - 1\right) \tag{72}$$

and z is positive downward. Now from equation (29),

$$\frac{\partial\theta}{\partial z} = \frac{d\theta_o}{dz} - \frac{t/c}{1 + \delta J_o t/cz} \frac{d}{dz}\left(\frac{J_o}{z}\right) \tag{73}$$

and

$$\frac{dJ_o}{dz} = \frac{2}{\alpha}\frac{d\alpha}{dz} J_o + \delta J_o \frac{d\theta_o}{dz} \tag{74}$$

by the flux scale relation (56) with ω equal α. Substitution of (74) into (73) and elimination of t with (29) gives

$$\frac{\partial\theta}{\partial z} = \frac{d\theta_o}{dz} e^{\delta(\theta-\theta_o)} + \frac{\left(1-e^{\delta(\theta-\theta_o)}\right)}{\delta z}\left(1 - 2\frac{z}{\alpha}\frac{d\alpha}{dz}\right) \tag{75}$$

Then substitution of (75) into (71) yields the pressure head gradient as a function of θ:

$$\frac{\partial h}{\partial z} = -\frac{h_m}{\alpha}\left(\frac{1}{\alpha}\frac{d\alpha}{dz}\right) + \frac{a_m}{\alpha}\rho\, e^{\rho(\theta-\phi)} \tag{76}$$

$$\left[\frac{d\theta_o}{dz} e^{\delta(\theta-\theta_o)} + \frac{\left(1 - e^{\delta(\theta-\theta_o)}\right)}{\delta z}\left(1 - 2\frac{z}{\alpha}\frac{d\alpha}{dz}\right)\right]$$

for $z > 0$ downward. The gradient (76) neglects the gradient of c. With the flux given by (28), the conductivity is then given by

$$K(\theta,z) = J_o\, e^{\delta(\theta-\theta_o)} / (1 + \partial h/\partial z). \tag{77}$$

An advantage of this method is that conductivity is not restricted to a simple exponential model. On the other hand, a disadvantage is the required evaluation of the gradients $d\theta_o/dz$ and $d\alpha/dz$, since small errors in these gradients can easily yield nonsense results such as negative conductivity. The formula (77) gives most reasonable results when the gradients are estimated by their average values over the profiles. A suggested procedure is to estimate $d\theta_o/dz$ and $d\alpha/dz$ from the slope of linear regressions over the profile. Conductivity estimates can be further improved by using the actual α obtained from direct scaling of the pressure head, when available.

It is notable that if the pressure head gradient (76) is replaced by its average over scales α, then the conductivity (77) is scaled with the flux scales of (56).

Conductivity based on scaled drainage flux

An improved method for simultaneously estimating and scaling conductivity is obtained by applying the new scale relations (i) through (v). The method does not require detailed measurements of the hydraulic gradient over the entire soil profile; however, it does require that conductivity scales equal drainage flux scales ω. Furthermore, if pressure head scales α are assumed to equal flux scales, then measurement of a soil-water characteristic at a single location within a field is sufficient to apply the method.

The following method is based on application of the integral equation (66) associated with indirect scaling of conductivity. Hydraulic conductivity and flux are assumed to satisfy the exponential models (31) and (28), respectively. The water content profiles are assumed to satisfy the drainage equation (29) with δ common to all locations, so that the flux scales ω are given by (56) and (57). Then assuming that conductivity scale factors equal ω for each location gives

$$\frac{K(\theta,z)}{J(\theta,z)} = \frac{J_o}{K_o} e^{(\delta-\beta)(\theta-\theta_o)} = \frac{J_m}{K_m} e^{(\delta-\beta)(\theta-\bar{\theta}_o)} \tag{78}$$

The parameters $A = J_m/K_m$ and $B = \delta - \beta$ are common to all locations. Under these assumptions, the integral equation (66) becomes

$$A \int_{z'}^{z''} e^{B(\theta-\bar{\theta}_o)} dz = z'' - z' + h'' - h' + \varepsilon \tag{79}$$

where ε represents error. This error is due to measurement error and the fact that (66) need not constitute an equality when the various models for flux, conductivity, and pressure head are substituted into the integral equation. Indeed, the soil-water properties represent only approximations of actual values for the experimental range. An estimate of A and B is then obtained from least squares fit of (79) to the measured water-content profile and the pressure head at the two depths z' and z'', for all experimental times. That is, the sum of squares of error in the hydraulic head difference between z' and z'' over all experimental times for all soil profiles is minimized. If direct measurements of the pressure head are not available, h can be estimated from a characteristic given by (72) and the drainage scales with

$$h = h_m/\omega. \tag{80}$$

Using the known parameters J_o and δ from the scaled drainage profiles and the least squares estimates of A and B, the conductivity parameters are obtained for each location from

$$K_o = \frac{J_o}{A} e^{B(\bar{\theta}_o-\theta_o)} \quad \text{and} \quad \beta = \delta - B. \tag{81}$$

This method can also be applied independently to each soil profile (experimental plot). In such application, the conductivity and flux need not scale over locations, that is β and δ need not be common to all locations, provided only that A and B are common in a statistical sense.

Solution of the drainage equation

The drainage equation

$$J_o e^{\delta(\theta-\theta_o)} = K_o e^{\beta(\theta-\theta_o)} (1 + \partial h/\partial z) \tag{82}$$

coupled with the pressure head model

$$h = a (e^{\rho(\theta-\phi)} - 1) \tag{83}$$

constitutes a first order differential equation solvable for θ. A computer program for the Runge-Kutta solution of (82) was developed, which can simulate the water-content

profile for known flux and conductivity model parameters. This program can be applied to study layer effects as characterized by different scale factor distributions over the soil profile (scale heterogeneous soil).

Millington-Quirk Conductivity and Scaling

Estimates of conductivity, which were calculated by the instantaneous profile method applied to the 1-m plots, are restricted to a limited range of water content. Indeed, the measured water content within the 1-m plots seldom decreased below a value of 0.3 even for 60 days of drainage. Accurate calculations of conductivity for smaller water-content values are difficult, because the change in θ over long periods is comparable to measurement error. Thus, for smaller water contents, another method is needed to estimate conductivity. Such an extension of conductivity, which is compatible with the scale relations, is provided by the Millington and Quirk (1959) method.

Introduction

The Millington-Quirk formula for hydraulic conductivity is given by

$$K(\theta)_i = \frac{\rho g}{8\eta} \frac{\theta^p}{n^2} \sum_{j=i}^{n} (2j + 1 - 2i)\, r_j^2 \tag{84}$$

where $r_1, \ldots, r_n$ are the radii of pores corresponding to n pore classes. Millington and Quirk take n to be the total number of pore classes and $p = 4/3$, and $K(\theta)_i$ denotes the conductivity at a water content θ for which the pore classes with radii $r_i, r_{i+1}, \ldots, r_n$ are water filled.

In terms of the suction head h (negative of pressure head) corresponding to pores of radius r remaining filled at suction h, the radius is given by the capillary equation

$$r = 2\gamma/\rho g h. \tag{85}$$

Using (85) the conductivity becomes

$$K(\theta)_i = \frac{\gamma^2}{2\rho g \eta} \frac{\theta^p}{n^2} \sum_{j=i}^{n} (2j + 1 - 2i)/h_j^2 \tag{86}$$

where $\gamma^2/2\rho g\eta = 2.7 \times 10^2$ cm/sec at 20 °C. The sum in equation (86) begins with the suction head h_i corresponding to the radius of largest water filled pores and is over all pore classes when θ equals the saturated water content (porosity).

The conductivity model (84) is derived from Poiseuille's equation for flow through a narrow tube of radius r, and for i equal 1 the sum

$$\frac{\phi\pi}{n^2} \sum_{j=1}^{n} (2j - 1) r_j^2 \tag{87}$$

represents an average cross sectional flow area of the connecting pore tubes, where ϕ is the porosity. An average area (87) is obtained by counting the number of effective pore-to pore connections for n classes, and the flow area of connecting pores is that of the smaller pore or neck of contact. The average (87), which was derived by Marshall (1958),

assumes an effective pore area fraction per unit area of section equal to ϕ. Millington-Quirk estimated that fraction to be $\phi^{2/3}$. Substitution of an average radius r corresponding to (87) into Poiseuille's equation for the effective flow velocity through the pore volume gives p = 2 for Marshall's method and p = 4/3 for the Millington-Quirk method. In the Marshall method, however, the n included in the term θ^p/n^2 is replaced by (n − i + 1), which is the number of water-filled pore classes up to θ. Thus the particular power p used in these methods is determined by the particular estimate of effective pore flow area and pore interaction (continuity of pore contact).

Both Marshall and Millington-Quirk methods estimate h_j corresponding to pore radius r_j by dividing the water range 0 to ϕ into n equal intervals, and the h_j equal the suction head evaluated at the midpoint of the water content intervals. Then

$$h_j = h(\bar{\theta}_j) \qquad (j = 1, \ldots, n) \tag{88}$$

where

$$\bar{\theta}_j = (2n - 2j + 1)\,\phi/2n \tag{89}$$

and

$$h_1 < h_2 < \ldots < h_n.$$

The actual pore distribution is unknown and the uniform distribution implied by (88) and (89) is assumed. A matching factor, K_s/K_{sc}, is introduced to correct the calculation error inherent in the assumptions of a capillary flow model with pore radii corresponding to uniformly distributed water content (89). Here K_s is the actual conductivity at saturation and K_{sc} is the value calculated from equation (86). Conductivity is then given by

$$K(\theta)_i = \left(\frac{K_s}{K_{sc}}\right)\left(\frac{\gamma^2}{2\rho g \eta}\right)\frac{\theta^p}{n^2}\sum_{j=i}^{n}(2j + 1 - 2i)/h_j^2. \tag{90}$$

Modifications to Millington-Quirk conductivity

The pore radius corresponding to suction head given by (88) can become arbitrarily large for water-content values near saturation, when the number of pore classes n is large. Physically realizable radii of pores are actually restricted by an upper limit to pore size. Thus, a critical value of water content less than saturation must occur for which the pore radius determined by (88) and (89) cannot be greater than

$$r_c = 2\gamma/\rho g h(\theta_c), \tag{91}$$

where θ_c denotes this critical value. For a sample drying from saturation, θ_c is the lowest value of θ for which all pores remain filled, and all pore classes contain water when $\theta_c < \theta \leq \phi$. The critical value θ_c is the extreme limit of the capillary fringe of a soil-water characteristic curve. The summation in equation (90) is therefore terminated for $\bar{\theta}_j$ of (89) greater than some θ_c, which is to be estimated from the soil-water characteristic curve. In a similar way, the sum in equation (90) is truncated for pore classes with radii below some physical limit size. This lower limit of pore radii is considered as corresponding to θ for an air-dry soil.

Matching factor

The matching factor K_s/K_{sc} can be estimated from any measured value of conductivity

for water contents other than saturated. Let $f = K_s/K_{sc}$ and $\hat{K}(\theta)_i$ denote conductivity calculated with equation (86). The measured conductivity $K(\theta)$ is described by a model function which is least squares fitted to measurements in the interval θ_{min} to θ_o. A best fit estimate of the matching factor f for n pore classes is obtained by minimizing the sum of squares of errors ε_i given by the following:

$$\ln K(\theta_i) = \ln f + \ln \hat{K}(\theta)_i + \varepsilon_i \tag{92}$$

for $\theta_{min} \leq \theta_i \leq \theta_o$. Minimization yields

$$\ln f = \frac{1}{N} \sum_{i=1}^{N} \ln(K_i/\hat{K}_i) \tag{93}$$

where N is the number of θ_i values between θ_{min} and θ_o.

Suction head required in equation (86) is computed from the model soil-water characteristic curve fitted to measured pressure head and water content. A model function for $h(\theta)$ allows extrapolation beyond the experimental range as required in order to compute conductivity for low water contents not attained in the field experiment.

Computer program for Millington-Quirk conductivity

A special computer program was developed to compute conductivity from equation (90) with the following features:

(i) The summation in equation (90) is truncated for pore classes with pore radii larger or smaller than predetermined physically realistic pore size limits. These pore size limits are determined by limits on the corresponding water content.

(ii) The best fit model soil-water characteristic curve

$$h(\theta) = a(e^{\rho(\theta-\phi)} - 1) \tag{94}$$

is used to estimate suction head and extrapolate beyond the range of experimental water content.

(iii) The best fit matching factor, $f = K_{act}/K_{calc}$, is computed for the model conductivity function

$$K(\theta) = K_o\, e^{\beta(\theta-\theta_o)} \tag{95}$$

fitted to experimental values in the range $\theta_{min} \leq \theta \leq \theta_o$.

(iv) An extended model function given by

$$K(\theta) = K_o \exp\left[\sum_{k=1}^{P} \beta_k\, (\theta-\theta_o)^k\right] \tag{96}$$

is least squares fit to the conductivity computed from equation (90), using the best fit matching factor of (iii). The fit of (96) includes the optional constraint that β_1 equals β. Then (96) is consistent with measured K for θ near θ_o. The program computes $\beta_1, \ldots, \beta_P$ step-wise with polynomial degree P incre-

mented by one until a desired accuracy of fit is reached. As a result of this method, the conductivity model (95) is extended to low water contents in a way compatible with field measurement and the Millington-Quirk model.

Scaling

The scaling of Millington-Quirk conductivity follows directly from the scaling of the soil-water characteristics. This result is demonstrated by letting α be the scale factor for any location where $\alpha h = h_m$ and by assuming that the matching factor has a fixed value. Conductivity is computed from equation (90) for each location, and the scale mean conductivity is defined by

$$K_m(\theta)_i = \left(\frac{K_s}{K_{sc}}\right)\left(\frac{\gamma^2}{2\rho g\eta}\right)\frac{\theta^p}{n^2}\sum_{j=i}^{n}(2j + 1 - 2i)/h_m^2(\bar{\theta}_j) \quad (97)$$

for $\bar{\theta}_j$ ($j = 1, \ldots, n$) given by (89). Then

$$K(\theta)_i = \alpha^2 K_m(\theta)_i. \quad (98)$$

The exact scaling relationship (98) is a consequence of the scaling of the capillary equation and Poiseuille's equation, which is just a special case of Stokes' equation.

Differences between the conductivity scales ω computed directly from measurements at each location and the pressure head scales α can be attributed to spatial variation in the estimated matching factor. Variations in the actual pore size distribution used to define the summation in equation (90) for each location can also cause deviations from the ideal scale relation (98).

Since porosity usually changes with location, best scaling results are obtained when the pressure head is expressed as a function of the saturation variable $s = \theta/\phi$. In terms of saturation, the Millington-Quirk formula becomes

$$K(s)_i = \left(\frac{K_s}{K_{sc}}\right)\left(\frac{\gamma^2}{2\rho g\eta}\right)\phi^p\frac{s^p}{n^2}\sum_{j=i}^{n}(2j + 1 - 2i)/h_j^2 \quad (99)$$

with $\bar{s}_j = \bar{\theta}_j/\phi$ and $h_j = h(\bar{s}_j)$. Now if ϕ^p does not vary appreciably over locations, which is usually the case, then conductivity given by (99) scales with the pressure head scale factor α according to the relation

$$K(s)_i = \alpha^2 K_m(s)_i \quad (100)$$

where the scale mean conductivity is defined by using (99) with $h_m(s)$. It is observed that conductivity can be estimated for all locations if the matching factor can be estimated for at least one location. This result assumes, of course, that the pressure head can be scaled and the pore size distributions for each location are similar. Indeed, the latter requirement is valid for locations having similar soil.

Average Drainage Flux

The effects of uncertainty in measurements of water content and conductivity have not been considered in any of the foregoing analyses. It was a tacit assumption in the previous sections that soil-water variables θ, h, and K and model function parameters represent mean values to be used in the Richards' equation. In this section, the view is

taken that Richards' equation is stochastic, because the soil-water variables are stochastic. The effect of both local and spatial variability on estimation of flux will now be considered. Random behavior of conductivity results from measurement error and fluctuation phenomena inherent in soil-water transport, and this random behavior propagates as statistical error in the water-content profiles.

Local variation of conductivity

The conductivity is described effectively with a regression model (32), which assumes normal distribution of errors in ln K and homogeneous variances within each location for a normal distribution of errors in θ. Conductivity is scaled by obtaining an estimate of β common to all locations under consideration, but K_0 and θ_0 depend on the location. In the following analyses of error propagation, β is assumed to be estimated with negligible error by the scaling method. Then conditional on this value of β, a single measurement of conductivity K for a value of θ is sufficient to estimate a value of K_0 from (32). The expectation of ln K_0 for these estimates is denoted by ln $\hat{K}_0$ where $\hat{K}_0$ represents the geometric mean for each location. That is,

$$\ln \hat{K}_0 = E[\ln K_0] \tag{101}$$

for each location. Now neglecting errors in β and θ_0 the standard deviation of ln K_0 is given by

$$\sigma[\ln K_0] = \sigma[\ln K] \tag{102}$$

where σ[ln K] is the standard error of estimate for the regression

$$\ln \hat{K} = \ln \hat{K}_0 + \beta(\theta-\theta_0). \tag{103}$$

The standard error of estimate of the mean ln $\hat{K}_0$ is estimated as

$$\sigma[\ln \hat{K}_0] = \sigma[\ln K_0]/\sqrt{N} \tag{104}$$

where N is the number of conductivity measurements for the location. Notice that (102) and (104) attribute a maximum error to ln $\hat{K}_0$. Estimates of the arithmetic mean $\overline{K}_0$ and standard deviation σ_{K_0} are obtained by transformation to a log-normal distribution as follows:

$$E[K_0] = \hat{K}_0 \exp(\sigma^2[\ln K_0]/2) \tag{105}$$

and

$$\sigma[K_0] = E[K_0] \left(\exp(\sigma^2[\ln K_0]) - 1\right)^{1/2}. \tag{106}$$

The error in $\overline{K}_0$ is

$$\sigma[\overline{K}_0] = \sigma[K_0]/\sqrt{N}. \tag{107}$$

As indicated in Part I both the arithmetic mean conductivity and standard error depend exponentially on θ.

Flux as a stochastic function

Prediction of flux for conditions of unit hydraulic gradient will now be investigated. Since K_0 can be viewed as a random variable which reflects the variability of the conductivity (31), the drainage equation (25) is a stochastic differential equation. Furthermore, the flux is a stochastic function of the random variable K_0, characterized by the statistics

(105) and (106). The average water-content profile, $\bar{\theta}(z,t)$, is obtained by averaging the solutions of equation (25) over the distribution of K_o. It is an approximation to assume that $\bar{\theta}(z,t)$ satisfies the same drainage equation (26) for the average $\bar{K}_o$.

. Random variations of K_o occur between locations (spatial) due to changes in soil properties such as texture and composition and within locations (local) due to fluctuation in transport phenomena and measurement errors. Therefore, the water content $\theta(z,t)$ must be averaged both between and within locations in order to compute the averages of cumulative seepage and flux through a field at any time. Variations in K_o can also be due to progressive deviation of measured conductivity and the exponential model (31), but this effect caused by lack of fit will be neglected. Hence, the local component of variability is represented by the standard error or estimate of conductivity and the spatial component by the scale distribution for the mean $\bar{K}_o$. Note that the mean steady state conductivity, $\bar{K}_o$, expressed relative to a common average $\bar{\theta}_o$ in the scale relation (38) is the quantity $\bar{\bar{K}}_o$. That is, in previous sections K_o represented a mean value; here K_o represents a single measurement.

Average cumulative seepage

Cumulative seepage at depth z (positive) equals $z\Delta(K_o)$ where

$$\Delta(K_o) = \hat{\theta}_o - \hat{\theta} = \frac{1}{\hat{\beta}} \ln[1 + \hat{\beta} K_o t/z]. \tag{108}$$

$\hat{\beta}$ is constant over locations if the variation in c is neglected, and the water content difference (108) is a stochastic function of the random variable K_o.

Local expected seepage

The local variation in K_o will be considered first. Denote the expectation of (108) as

$$\bar{\Delta} = E[\Delta(K_o)]. \tag{109}$$

A second order Taylor series expansion about the mean $\bar{K}_o$ gives

$$\Delta(K_o) = \Delta(\bar{K}_o) + \Delta'(\bar{K}_o)(K_o - \bar{K}_o) + \frac{\Delta''}{2}(\bar{K}_o)(K_o - \bar{K}_o)^2. \tag{110}$$

Taking the expectation of (110) yields the following approximation:

$$\bar{\Delta} = \Delta(\bar{K}_o) + \frac{1}{2}\Delta''(\bar{K}_o)\,\sigma^2_{K_o}. \tag{111}$$

Similarly, a first order approximation of the standard deviation is given by

$$\sigma[\Delta] = |\Delta'(\bar{K}_o)|\,\sigma_{K_o}. \tag{112}$$

The derivatives required in the equations (111) and (112) are

$$\Delta'(K_o) = \frac{t/z}{1 + \hat{\beta} K_o t/z} \tag{113}$$

and

$$\Delta''(K_o) = -\hat{\beta}\left(\frac{t/z}{1 + \hat{\beta} K_o t/z}\right)^2 = -\hat{\beta}[\Delta'(K_o)]^2. \tag{114}$$

Equation (111) demonstrates explicitly that the local average depends on the sample

variance of K_o as well as on the mean of K_o. For times sufficiently later than initial drainage, equations (111) and (112) simplify to

$$\bar{\Delta} = \Delta(\bar{K}_o) - \frac{1}{2\hat{\beta}} (\sigma_{K_o}/\bar{K}_o)^2 \tag{115}$$

and

$$\sigma[\Delta] = \frac{1}{\hat{\beta}} (\sigma_{K_o}/\bar{K}_o). \tag{116}$$

Equation (111) with (114) indicates that the expected seepage is always less than the deterministic estimate obtained by substituting $\bar{K}_o$ into equation (108). Average flux within a location is given by

$$\bar{J} = z \frac{\partial \bar{\Delta}}{\partial t}. \tag{117}$$

Thus the average flux does not equal the deterministic estimate.

Spatial variation in seepage

Next, consider the spatial variation in K_o. $\bar{\Delta}_r$ will denote the mean water-content difference at each location r, according to equation (111). The spatial average of (108) is

$$\bar{\Delta} = \frac{1}{R} \sum_{r=1}^{R} \bar{\Delta}_r \tag{118}$$

and spatial variance is

$$\nu^2 = \frac{1}{R-1} \sum_{r=1}^{R} (\bar{\Delta}_r - \bar{\Delta})^2. \tag{119}$$

Variance in the spatial average (118) due to variation within locations is

$$\sigma^2[\bar{\Delta}] = \frac{1}{R^2} \sum_{r=1}^{R} \sigma^2[\bar{\Delta}_r] \tag{120}$$

where $\sigma[\bar{\Delta}_r]$ is the standard error of the mean $\bar{\Delta}_r$, i.e., standard deviation $\sigma[\Delta_r]$ divided by the square root of sample size. The variance (120) assumes independent distributions of Δ_r for all locations. The avarage local error equals

$$\frac{1}{R} \sum_{r=1}^{R} \sigma[\bar{\Delta}_r]. \tag{121}$$

Using a second order expansion, the spatial average (118) is given approximately by

$$\bar{\Delta} = \Delta(\bar{K}_o) + \frac{1}{2} \Delta''(\bar{K}_o) \left[\nu_o^2 + \frac{1}{R} \sum_{r=1}^{R} \sigma_{K_{or}}^2 \right] \tag{122}$$

where

$$\bar{K}_o = \frac{1}{R} \sum_{r=1}^{R} \bar{K}_{or} \quad \text{and} \quad \nu_o^2 = \frac{1}{R-1} \sum_{r=1}^{R} (\bar{K}_{or} - \bar{K}_o)^2. \tag{123}$$

The term in the brackets of (122) is recognized as the sum of spatial and average local variance of K_o. It can be shown that the spatial variance (119) has the following approximation:

$$\nu^2 = \frac{1}{R-1} \sum_{r=1}^{R} [\Delta'(\bar{K}_o) + \frac{1}{2}(\Delta')'(\bar{K}_o)\, \sigma^2_{K_{or}}]^2 (\bar{K}_{or} - \bar{K}_o)^2. \quad (124)$$

Replacement of the local variances in equation (124) by average local variance yields a further simplified approximation:

$$\nu^2 = \left[\Delta'(\bar{K}_o) + \frac{1}{2}(\Delta')'(\bar{K}_o) \left(\frac{1}{R} \sum_{r=1}^{R} \sigma^2_{K_{or}}\right)\right]^2 \nu_o^2. \quad (125)$$

Equation (125) indicates that the spatial variance depends on the local variance of K_o as well as on the spatial variance of K_o.

The total variance of K_o is

$$\sigma^2 = \nu_o^2 + \frac{1}{R} \sum_{r=1}^{R} \sigma^2_{K_{or}}. \quad (126)$$

And according to equation (122) the complete spatial average of cumulative seepage equals

$$\frac{z}{\hat{\beta}} \ln[1 + \hat{\beta}\bar{K}_o t/z] - z\hat{\beta} \left(\frac{t/z}{1 + \hat{\beta}\bar{K}_o t/z}\right)^2 \sigma^2. \quad (127)$$

The average flux equals the derivative of (127) with respect to time.

Although (127) is not an exact estimate of average cumulative seepage, it does demonstrate explicitly the departure from a deterministic estimate using only the mean $\bar{K}_o$. It is noted that the above derivations apply as well to the actual drainage described by equation (29). The parameters need only be changed to $\hat{\delta}$ and J_o.

Estimate of average flux

A first order expansion of the flux for unit hydraulic gradient conditions can be used to investigate the effect of variance of K_o on average flux. Here $\sigma^2_{K_o}$ will represent either the local or spatial variance. A first order expansion of flux about the mean $\bar{K}_o$ gives

$$J = \frac{K_o}{1 + \hat{\beta}\bar{K}_o t/z} \left[1 - \frac{\hat{\beta}(K_o - \bar{K}_o)t/z}{1 + \hat{\beta}\bar{K}_o t/z}\right]. \quad (128)$$

Then the expectation of flux is approximately given by

$$\bar{J} = \frac{\bar{K}_o}{1 + \hat{\beta}\bar{K}_o t/z} \left[1 - \frac{\hat{\beta}t/z}{1 + \hat{\beta}\bar{K}_o t/z} \left(\frac{\sigma^2_{K_o}}{\bar{K}_o}\right)\right]. \quad (129)$$

Now for $\hat{\beta}\bar{K}_o t/z << 1$, the equation (129) becomes approximately

$$\bar{J} = \frac{\bar{K}_o}{1 + \hat{\beta}\bar{K}_o(1 + (\sigma_{K_o}/\bar{K}_o)^2)t/z} . \tag{130}$$

Thus the average flux is given by the deterministic equation for the parameter $\hat{\beta}$ effectively increased by a factor depending on the square of the coefficient of variation of K_o. When (130) represents a local average flux, equation (130) is consistent with the direct fit of the drainage equation (29), since δ is greater than β for all locations. Including the local variation of K_o evidently compensates partially for the incorrect assumption that $\partial h/\partial z = 0$, by effectively decreasing conductivity for each value of θ. Indeed, $\partial h/\partial z$ is negative for the 1-m plots during drainage, so that numerically flux is less than conductivity.

Spatial distribution of flux

The spatial distribution of K_o is usually log-normal because the scale factors α are found to be distributed in that way. This result is a consequence of the scale relation:

$$\ln K_o = 2 \ln \alpha + \ln K_m . \tag{131}$$

Equation (131) includes the assumption that θ_o is constant. The result still holds, however, if θ_o is normally distributed, and this is often the case. The distribution of $\ln J$ is deduced from

$$\ln J = \ln K_o - \ln(1 + \hat{\beta}K_o t/z) . \tag{132}$$

When t is sufficiently large,

$$\ln J = \ln(\hat{\beta}t/z) \tag{133}$$

and the flux reduces to a single value determined by the common parameter $\hat{\beta}$ obtained from scaling, provided the variation of the regression coefficient c is negligible. In general, the distribution of $\ln J$ eventually assumes the distribution of $\ln c$. Initially for $t = 0$, the flux is log-normally distributed, being identical to K_o. For sufficiently small t, (132) becomes approximately

$$\ln J = \ln K_o - \hat{\beta}K_o t/z . \tag{134}$$

Thus when (134) holds, flux is approximately log-normally distributed with a reduced skewness. Standard deviation for the distribution described by (134) is given approximately by

$$\sigma[\ln J] = |1 - \hat{\beta}\hat{K}_o t/z| \; \sigma[\ln K_o] \tag{135}$$

where $\hat{K}_o$ denotes the geometric mean. Therefore for sufficiently small t such that (134) is valid the standard deviation of $\ln J$ is reduced as time progresses.

The asymptotic behavior of flux indicated by (133) and (134) implies that the distribution of $\ln J$ corresponding to spatial variation of K_o will be cut off at large K_o values. Thus values of K_o greater than a certain cut-off value given by

$$K_c = 2z/\hat{\beta}t \tag{136}$$

will contribute little to the extreme end of the flux distribution. Although the above concerns the distribution of flux for unit hydraulic gradient, the conclusions also apply to the actual flux described by equation (30). This is so because the drainage scale factors

ω of (56) are also log-normally distributed. Indeed, for the 1-m plots the scales α and ω are essentially equal for all locations.

In contrast to flux, the cumulative seepage begins without variation and eventually assumes the distribution of ln α. That is, after sufficiently long time the cumulative seepage equals

$$\frac{z}{\hat{\beta}} \ln(\hat{\beta} K_m t/z) + \frac{2z}{\hat{\beta}} \ln \alpha. \tag{137}$$

Thus, ultimately the spatial variability of the soil locations is reflected by the measured variability of the cumulative seepage. Again the variability of θ_o has been neglected in (137). In view of the scale relation (38) and (35), in general the term $zc(\theta_o - \bar{\theta}_o)$ must be added to (137).

Exact statistics of seepage and flux

The above discussion applies only to the asymptotic behavior of the flux distribution for scaled drainage profiles. In this section, an exact treatment which requires direct computation of the statistics of (132) from the distributions of K_o and c is provided. Previously, the variation of the regression coefficient c was neglected. Here the combined variation of c and K_o is taken into account. Scaling of the soil properties is assumed, so that β is common to all locations. Again these results will apply to the actual measured drainage scaled with equation (29) as well as for unit hydraulic gradient condition.

Another random variable defined by $k = K_o/c$ is log-normally distributed if both K_o and c are log-normally distributed. The probability distribution of k will be denoted by P(k). Expected cumulative change in water content is

$$E[\Delta] = \int \Delta(k)\, P(k)\, dk \tag{138}$$

where

$$\Delta(k) := \theta_o - \theta = \frac{1}{\beta} \ln[1 + \beta k t/z], \tag{139}$$

and the expected cumulative seepage equals

$$zE[c]\, E[\Delta]. \tag{140}$$

The average seepage (140) assumes that c and Δ are independent random variables, which is approximately the case if K_o and c are independent. It should be noted when applying (138) that the simple drainage equation (139) remains valid only for t such that $\Delta \leq \theta_o$. The probability distribution of the cumulative change Δ is

$$P(\Delta) = \frac{d}{d\Delta} \int_0^{k(\Delta)} P(k')dk' \tag{141}$$

where

$$k(\Delta) = z(e^{\beta\Delta} - 1)/\beta t. \tag{142}$$

That is, (142) is the value of k that corresponds to Δ and satisfies equation (139). The integral in (141) represents the cumulative probability for a value of Δ.

Cumulative seepage is distributed as the independent product of c and Δ. Since the distribution of c is narrow, seepage has essentially the distribution of $z\bar{c}\Delta$. Variance of

Δ is

$$\sigma_{\Delta}^2 = E[\Delta^2] - E[\Delta]^2, \tag{143}$$

and variance of cumulative seepage equals

$$z^2\{\sigma_c^2\,\sigma_{\Delta}^2 + \bar{c}^2\,\sigma_{\Delta}^2 + \bar{\Delta}^2\,\sigma_c^2\}. \tag{144}$$

The expected logarithm of flux is

$$E[\ln J] = E[\ln K_o] - \beta E[\Delta] \tag{145}$$

where

$$E[\ln K_o] = E[\ln k] + E[\ln c]. \tag{146}$$

and variance satisfies

$$\text{var}[\ln k] = \text{var}[\ln K_o] + \text{var}[\ln c]. \tag{147}$$

The logarithm of flux is distributed as the difference, $\ln K_o - \beta\Delta$, or $\ln J - \ln c$ is distributed as $\ln k - \beta\Delta(k)$. Variance of $\ln J$ is

$$\text{var}[\ln J] = \text{var}[\ln k - \beta\Delta(k)] + \text{var}[\ln c]. \tag{148}$$

The arithmetic mean of flux and variance σ_J^2 can be computed as follows. Mean flux is

$$\bar{J} = z\bar{c}\,\frac{\partial\bar{\Delta}}{\partial t} \tag{149}$$

and variance is

$$\sigma_J^2 = z^2\{\sigma_c^2\,\nu^2 + \bar{c}^2\,\nu^2 + \left(\frac{\partial\bar{\Delta}}{\partial t}\right)^2\sigma_c^2\} \tag{150}$$

where

$$\nu^2 = \text{var}\left[\frac{\partial\Delta}{\partial t}\right]. \tag{151}$$

Usually, the distribution of k for each location r is found to be log-normal:

$$P_r(k) = \frac{1}{\sqrt{2\pi}\,\sigma_r k}\exp[-(\ln k - \mu_r)^2/2\sigma_r^2] \tag{152}$$

where μ_r and σ_r^2 denote the mean and variance of $\ln k$, respectively. This is the local distribution. The index r can specify a collection of locations occurring with frequency w_r in a particular population of similar soil locations. A pooled distribution representing both local and spatial variability for R locations is then given by

$$P(k) = \sum_{r=1}^{R} w_r P_r(k). \tag{153}$$

It is the probability distribution (153) that determines the statistics of the random functions of k, i.e., of steady state conductivity.

The utility of scaling is now evident. Averages of cumulative seepage and flux can be computed with respect to a single parameter rather than require a multi-variate average as discussed by Warrick *et al.* (1977).

Spatial Extension of Conductivity

Before the various formulae for average seepage and flux can be applied to the experimental field, it is necessary to ascertain conductivity at every location. Although conductivity was measured for only a few locations, it can be estimated for all locations by using similarity of soil properties. The scale factor distribution associated with scaling pressure head provides an extension of conductivity measured in the 1-m plots to the entire experimental field. All that is required by the procedure is an estimate of the scale mean conductivity and an assumption of equality of scale factors derived from pressure head and conductivity. This assumption is the principle of scaling and is verified for the 1-m plots, as well as for some other soils (Warrick, Mullen, and Nielsen, 1977). Conductivity is then generated by its scale relation.

Because the scale factors satisfy the constraint of average equal to unity, the magnitudes of scales and scale mean conductivity depend on the number of locations. This is demonstrated as follows. Consider the scaling of conductivity for a subset of locations. Let $\alpha_1, \ldots, \alpha_R$ denote a set of scale factors satisfying the scale relation

$$K_r = \alpha_r^2 K_m \qquad (r = 1, \ldots, R) \tag{154}$$

and the constraint

$$\frac{1}{R} \sum_{r=1}^{R} \alpha_r = 1. \tag{155}$$

Notice that the conductivity is not required to satisfy a simple exponential model. Now let K_{r_q} $(q = 1, \ldots, Q)$ with $Q < R$ denote a subcollection of K_r. New scales α_q' relative to a new scale mean K_m' can be computed for the Q locations. The new scales satisfy

$$K_{r_q} = \alpha_q'^2 K_m' \quad \text{and} \quad \frac{1}{Q} \sum_{q=1}^{Q} \alpha_q' = 1. \tag{156}$$

The scale means K_m and K_m' are related by the mean conductivity for the subcollection of locations:

$$\bar{K} = \frac{1}{Q} \sum_{q=1}^{Q} K_{r_q} = \frac{1}{Q} \sum_{q=1}^{Q} \alpha_{r_q}^2 K_m = \frac{1}{Q} \sum_{q=1}^{Q} \alpha_q'^2 K_m' \tag{157}$$

where

$$K_m'^{1/2} = \frac{1}{Q} \sum_{q=1}^{Q} K_{r_q}^{1/2}. \tag{158}$$

The new scales are related to the original α_r by

$$\alpha_q' = \alpha_{r_q} \Big/ \frac{1}{Q} \sum_{q=1}^{Q} \alpha_{r_q}. \tag{159}$$

Now if the average in the denominator of (159) is less than unity, then the new scales are greater than the originals for every location. Moreover, the scale mean conductivity

for the subcollection of locations is less than that of the entire set. The opposite holds if the denominator is greater than unity. Similar rescaling of the pressure head for a subset of locations again yields (159); however, the scale mean pressure head for the subset increases if K_m decreases, and conversely.

Equation (157) provides the basis for estimating the conductivity for a set of R locations from measurements on a smaller set of Q locations, when the scales $\alpha_1, \ldots, \alpha_R$ are known from the scaled soil-water characteristics. Furthermore, the two sets of locations need not overlap if the soil is similar over the combined set of locations.

Matching of scale distributions

A general method based on equation (157) for matching scale factor distributions and spatially extending conductivity will now be established.

Two different sets of scale factors will be denoted by α and α'. The number of scales in each set is denoted by N and N′, respectively, and these numbers correspond to the number of locations. Scales satisfy the normalization constraints:

$$\frac{1}{N}\sum \alpha = 1 \quad \text{and} \quad \frac{1}{N'}\sum \alpha' = 1.$$

Now let α_r (r = 1, . . . , R) and a'_q(q = 1, . . . , Q) with R ≤ N and Z ≤ N′ represent two subcollections of scales which correspond to conductivity measurements assumed taken from the same sample distribution. For example, the two subcollections could be scale factors from two different scaling experiments within similar soil, such as the experimental field plots and the 1-m plots, and associated with all locations at the same depth. Here conductivity measurements for a particular depth are assumed to be obtained from the same sample distribution (population) for a similar soil. This is a fundamental statistical assumption. The expected mean of the conductivity sample distribution is denoted by $\overline{K}$. Conductivity for each location is given by

$$K_r = \alpha_r^2 K_m \quad (r = 1, \ldots, R) \text{ and}$$

$$K'_q = \alpha'^2_q K'_m \quad (q = 1, \ldots, Q).$$

Now the means

$$K = \frac{1}{R}\sum_{r=1}^{R} K_r \quad \text{and} \quad K' = \frac{1}{Q}\sum_{q=1}^{Q} K'_q$$

are estimates of $\overline{K}$ with standard errors

$$\sigma_K = \nu/\sqrt{R} \quad \text{and} \quad \sigma_{K'} = \nu'/\sqrt{Q},$$

respectively, where

$$\nu^2 = \frac{1}{R-1}\sum_{r=1}^{R} (K_r - K)^2 \quad \text{and}$$

$$\nu'^2 = \frac{1}{Q-1}\sum_{q=1}^{Q} (K'_q - K')^2$$

are the sample variances. The scale mean conductivity K_m' can be estimated from the known scales and K_m by equating estimates of $\overline{K}$:

$$K = \frac{1}{R}\sum_{r=1}^{R} \alpha_r^2 K_m = \frac{1}{Q}\sum_{q=1}^{Q} \alpha_q'^2 K_m'. \tag{160}$$

Assuming a fixed distribution for the scale values α', an estimate of the error in K_m' is given by

$$\sigma_K = \frac{1}{Q}\sum_{q=1}^{Q} \alpha_q'^2 \sigma_{K_m'}. \tag{161}$$

The error in estimated conductivity K_q' at each location q is

$$\sigma_{K_q'} = \alpha_q'^2 \sigma_{K_m'}. \tag{162}$$

If in addition α_q' are subject to error, then the effect of this on K_q' must be included, and (162) is not valid in that case. Thus the scale distributions are matched and the proper scale mean conductivity K_m' is estimated relative to another set of scales α'. A method to estimate scale mean pressure head h_m' relative to another set of scales is based on the following equation:

$$\bar{h} = \frac{1}{R}\sum_{r=1}^{R} h_r = \frac{1}{Q}\sum_{q=1}^{Q} \frac{1}{\alpha_q'} h_m'. \tag{163}$$

Note that scale matching methods are not unique. Analogous methods based on definitions of an average other than the arithmetic mean can be constructed. This is expected, since there does not exist a single "best" way of defining scale factors.

Results and Discussion: Scaling

Hydraulic conductivity: finite difference estimates

Soil-water storage and flux for the four 1-m plots were computed for 15 cm intervals down to the 120 cm depth. Conductivity was calculated at the 60, 75, 90, 105, and 120 cm depths, using direct finite difference estimates of flux and hydraulic gradient based on pressure head differences between the 60 and 120 cm depths. The exponential model in its linear form (32) was least squares fit to the estimated conductivity for the 20 soil locations. Parameters for (32) and regression statistics are given in table 1. The regression was restricted to conduction of positive or downward flux, so that the logarithm has a defined value. This was necessary because the direct finite difference estimates of conductivity did not involve prior data smoothing. Such estimates of conductivity are expected to represent the largest possible measurement error, but are the least biased. The steady state water content (initial value) given in table 1 is the actual measured value, and the estimated steady state conductivity K_o (initial value) is a geometric mean value. As seen in table 1, generally both the parameters K_o and β increase with depth, and the standard errors of estimate of ln K are approximately homogeneous over locations. Correlation coefficients, which range between 0.68 and 0.82,

indicate that the exponential model adequately describes conductivity, within measurement error. Standard errors of θ_0 and β are indicated by parentheses. Typically, the initial value of water content θ_0 had the largest measurement error, and the error in θ ranged between 0.001 and 0.005 after 2 days of drainage.

The finite difference estimates of conductivity used in table 1 were scaled with the model (33) in terms of an approximate saturation variable θ/θ_0 and for s_0 equal 1 over all locations. Estimated parameters and scale factors determined relative to a common value of b are given in table 2. The parameter β for the exponential model (31) in terms of water content equals b/θ_0. Ideally, for perfect similarity, the parameters obtained by scaling would equal those obtained by an independent direct fit of the conductivity model for each location. Table 2 when compared to table 1 indicates that scaling of conductivity holds approximately. Although the pattern of variation of K_0 is different in tables 1 and 2, the standard error of estimate of ln K is only slightly increased

TABLE 1.

REGRESSIONS FOR HYDRAULIC CONDUCTIVITY EQUATION (31) FOR 20 LOCATIONS IN THE 1-M PLOTS. PARENTHESES INDICATE STANDARD ERRORS. σ DENOTES THE STANDARD ERROR OF ESTIMATE. SAMPLE SIZE IS N AND CORRELATION COEFFICIENT IS R. K_0 (CM/DAY) IS THE GEOMETRIC MEAN.

Depth cm	θ_0	K_0	β	σ[ln K]	N	R
			PLOT 1			
60	0.422 (.010)	2.80	22.3 (6.0)	0.782	16	0.70
75	0.415 (.014)	3.56	26.6 (6.4)	0.750	16	0.74
90	0.411 (.012)	4.50	31.8 (7.3)	0.784	16	0.76
105	0.409 (.004)	4.53	41.6 (9.1)	0.869	17	0.76
120	0.426 (.009)	5.21	46.9 (10.5)	0.868	16	0.77
			PLOT 2			
60	0.411 (.010)	2.55	24.8 (6.5)	0.856	14	0.74
75	0.412 (.012)	3.83	26.1 (6.3)	0.839	14	0.77
90	0.410 (.007)	4.84	29.8 (6.4)	0.794	15	0.79
105	0.414 (.007)	5.73	32.4 (6.7)	0.807	15	0.80
120	0.420 (.010)	6.66	35.3 (6.9)	0.805	15	0.82
			PLOT 3			
60	0.414 (.015)	0.94	33.3 (11.0)	1.058	16	0.63
75	0.421 (.020)	2.07	34.4 (8.8)	0.935	16	0.72
90	0.427 (.015)	3.43	37.8 (7.0)	0.811	16	0.82
105	0.428 (.013)	3.05	40.2 (8.6)	0.855	16	0.78
120	0.429 (.017)	3.47	33.1 (5.9)	0.772	17	0.82
			PLOT 4			
60	0.382 (.011)	3.23	25.3 (6.1)	0.771	15	0.75
75	0.385 (.010)	4.01	28.5 (6.7)	0.791	15	0.76
90	0.391 (.010)	4.97	36.5 (9.7)	0.914	15	0.72
105	0.394 (.006)	4.90	35.1 (10.4)	0.995	15	0.68
120	0.415 (.006)	5.35	36.6 (10.0)	1.009	15	0.71

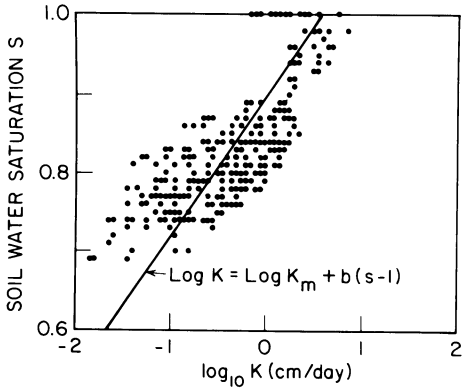

Fig. 1. Scaled conductivity for the 60, 75, 90, 105, and 120 cm depths of the 1-m plots. Degree saturation s equals θ/θ_0. Solid line represents the scale mean conductivity with parameters b = 13.0 and K_m = 3.79 cm/day.

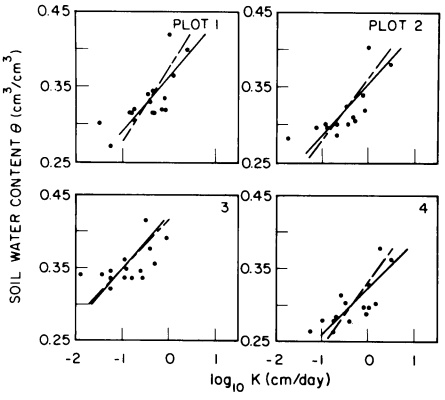

Fig. 2. Conductivity versus water content at the 60 cm depth in the 1-m plots. Curves are given by equation (32). Solid lines are based on table 2 and broken lines on table 1.

in table 2. The most important statistic here is the standard error of estimate, since it determines the accuracy of conductivity predicted by the model. Indeed, the uncertainty in the parameters K_0 and β caused by the considerable measurement error in conductivity allows a range of possible values for these parameters. The estimated value of b is 13.0 and the scale mean value of steady state conductivity K_m is 3.79 cm/day. Scale factors in table 2 obtained with the scale relation (38) correspond to an estimated geometric mean of K_0. Scaled conductivity for the entire 20 locations is shown in figure 1 where the solid line represents the scale mean value. Data in figure 1 were coalesced with the scale relation (38), that is, each conductivity measurement is divided by the squared scale factor of the locations. Pooled standard errors of estimate of ln K and degree saturation are 0.89 and 0.06, respectively.

Figures 2 and 3 depict the fit of the exponential model (31) for the 60 and 120 cm

TABLE 2.

PARAMETERS AND SCALE FACTORS FOR THE SCALED CONDUCTIVITY EQUATION (33) WITH s_0 = 1. θ_0 ARE THE VALUES IN TABLE 1. b = 13.0 AND K_m = 3.79 CM/DAY, WHERE K_0 IS THE GEOMETRIC MEAN.

Depth	PLOT 1				PLOT 2			
cm	K_0	β	σ[ln K]	ω	K_0	β	σ[ln K]	ω
60	5.79	30.8	0.835	1.236	4.65	31.6	0.894	1.107
75	5.09	31.3	0.765	1.159	6.18	31.5	0.865	1.277
90	4.45	31.6	0.784	1.083	5.62	31.7	0.797	1.218
105	2.73	31.8	0.903	0.848	5.29	31.4	0.808	1.181
120	2.24	30.5	0.941	0.769	4.74	31.0	0.818	1.118
	PLOT 3				PLOT 4			
60	0.84	31.4	1.059	0.469	6.42	34.0	0.830	1.301
75	1.58	30.9	0.940	0.645	5.72	33.8	0.809	1.228
90	1.95	30.4	0.843	0.718	4.14	33.3	0.918	1.044
105	1.66	30.4	0.893	0.662	4.36	33.0	0.996	1.073
120	2.81	30.3	0.778	0.861	3.81	31.3	1.020	1.003

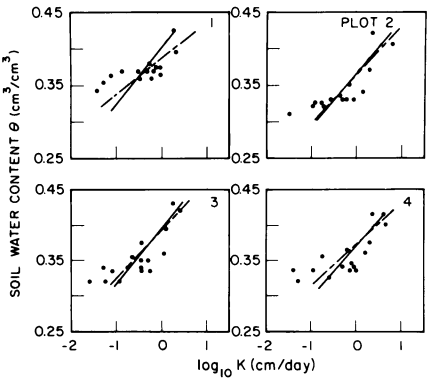

Fig. 3. Conductivity versus water content at the 120 cm depth in the 1-m plots. Curves are given by equation (32). Solid lines are based on table 2 and broken lines on table 1.

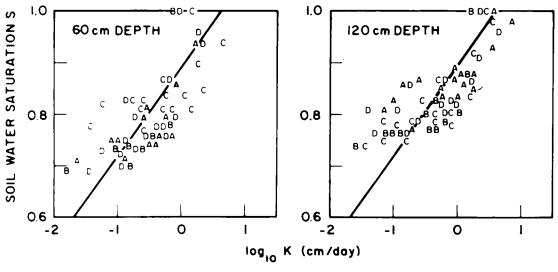

Fig. 4. Scaled conductivity for the 60 and 120 cm depths of the 1-m plots. Degree saturation equals θ/θ_0. Solid line represents the scale mean conductivity with parameters b = 13.0 and K_m = 3.79 cm/day. Letters A,B,C, and D denote measured values for the four plots.

depths, respectively. The broken line is determined by the parameters of table 1 and the solid line by those of table 2. Figure 4 shows the scaled conductivity in terms of approximate saturation for the 60 and 120 cm depths. In figures 2 through 4, the letters A, B, C, and D denote data for the four 1-m plots. The graphs shown in figures 2 and 3 indicate that conductivity is not exactly linear for semi-log coordinates, showing progressive deviation for lower water content values. This suggests that the simple exponential conductivity model requires correction terms at lower water content, as described by equation (96). However, accuracy of the present estimate of hydraulic head gradient used to calculate conductivity does not warrant an improved fit provided by such a complex model. Corrections to the estimation of conductivity deviates from an exponential model, the scale factors still describe the relative position of measured conductivity curves.

Estimates of arithmetic mean and standard deviation of K_0 along with corresponding scale factors are listed in table 3. These estimates are obtained by transformation to a log-normal distribution, assuming a homogeneous normal distribution of errors in ln K at each location. The method follows equations (105) and (106), and uses the standard error of estimate of ln K given in table 2. The scale mean K_m corresponding to the arithmetic means of table 3 is 5.53 cm/day, and is greater than the geometric mean. Coefficients of variation of K_0 are typically 100 percent of the mean value.

Uncertainty of measured conductivity also allows the possibility of scaling with the model (34) in terms of water content. The approximate constancy of β in table 2 indicates that the conductivity curves are similar with a common value of β. Scale factors calculated relative to water content and corresponding to arithmetic mean K_0 are given in table 4. For table 4, the scale mean K_m is 6.16 cm/day, and values of β and $\bar{\theta}_0$ common to all locations are 31.6 and 0.411, respectively. Calculation of these scale factors follows the approximate method given by equations (44) through (47), using the values of K_0 from table 3. Estimated error in the common β is 0.1. Scaling relative to a common $\bar{\theta}_0$ yields substantially different scale factors for plot 4. This is so because the θ_0 of plot 4 used to obtain the scales in table 3 are less than those for the other plots. That is, the θ_0 of plot 4 are not good estimates of water content under field saturated conditions. Apparently, since the data begins about one day after steady state ponding conditions, drainage in plot 4 was advanced beyond that of the other plots. The scale factors of table 4 attribute the advanced drainage to greater relative conductivity in

TABLE 3.

PARAMETERS AND SCALE FACTORS FOR THE SCALED CONDUCTIVITY EQUATION (33). θ_o ARE THOSE OF TABLE 1. b = 13.0 AND K_m = 5.53 CM/DAY. K_o (CM/DAY) IS THE ARITHMETIC MEAN.

Depth	PLOT 1			PLOT 2		
cm	K_o	$\sigma[K_o]$	ω	K_o	$\sigma[K_o]$	ω
60	8.21	8.24	1.218	6.93	7.67	1.120
75	6.82	6.08	1.110	8.98	9.48	1.274
90	6.05	5.58	1.046	7.72	7.27	1.181
105	4.10	4.61	0.861	7.33	7.04	1.151
120	3.49	4.16	0.794	6.62	6.46	1.094
	PLOT 3			PLOT 4		
60	1.47	2.12	0.516	9.06	9.02	1.280
75	2.46	2.93	0.667	7.93	7.63	1.198
90	2.78	2.83	0.709	6.31	7.26	1.068
105	2.47	2.73	0.669	7.16	9.33	1.138
120	3.80	3.47	0.829	6.41	8.67	1.076

TABLE 4.

SCALE FACTORS CORRESPONDING TO ARITHMETIC MEAN K_o FOR THE CONDUCTIVITY EQUATION (34) IN TERMS OF WATER CONTENT. COMMON PARAMETERS: β = 31.6, $\bar{\theta}_o$ = 0.411, AND K_m = 6.16 CM/DAY.

Depth	PLOT			
cm	1	2	3	4
60	0.980	1.067	0.468	1.997
75	0.994	1.195	0.544	1.770
90	0.997	1.143	0.529	1.419
105	0.847	1.047	0.492	1.435
120	0.601	0.908	0.602	0.964

TABLE 5.

SCALE FACTORS CORRESPONDING TO ARITHMETIC MEAN K_o FOR THE CONDUCTIVITY EQUATION (36) IN TERMS OF ACTUAL DEGREE SATURATION, AND NORMALIZED FOR THE 60 AND 120 CM DEPTHS. PARENTHETIC VALUES ARE SATURATED WATER CONTENT. COMMON PARAMETERS: b = 14.0, $\bar{s}_o$ = 1, AND K_m = 16.5 CM/DAY.

Depth	PLOT			
cm	1	2	3	4
60	1.083 (0.45)	1.020 (0.44)	0.448 (0.44)	2.230 (0.45)
120	0.665 (0.45)	0.868 (0.44)	0.763 (0.46)	0.922 (0.45)

plot 4. Table 4 provides the best description of relative spatial variability, since effects of different initial water contents θ_0 are removed. Moreover, when improved estimates of saturated water content ϕ are employed with the correction method of appendix F, the resulting scale factors for the model (36) in terms of actual degree saturation are comparable to those in table 4. Table 5 demonstrates this for the 60 and 120 cm depths for which ϕ is estimated from the measured soil-water characteristics. Actual ϕ for the other depths in the 1-m plot profile was not measured. Note that the value of K_m, 16.5 cm/day, is considerably greater than that of table 3, as a consequence of the decrease in degree saturation s. Estimates of conductivity, however, are not substantially changed, since equation (36) with $\bar{s}_0$ equal 1 is not defined beyond the saturation equal to θ_0/ϕ for each location. Indeed, the value of K_m can be decreased, without altering the scales, by defining (37) with $\bar{s}_0$ equal to an average of θ_0/ϕ.

Equality of pressure head and conductivity scale factors

Inasmuch as the pressure head was measured for only the 60 and 120 cm depths in the 1-m plots, a comparison of scale factors is restricted to these depths or 8 soil locations. Table 5 contains the scale factors for the conductivity model (36) in terms of actual degree saturation with $\bar{s}_0 = 1$. These scales correspond to arithmetic mean K_0, and are normalized for the eight locations. Conductivity scales ω in table 5 and pressure head scales α in table 1 of Part I are compatible with respect to normalization, and both models for those soil water properties are expressed in terms of actual degree saturation. Thus, the scale factors are now in representations that are compatible for a proper comparison. Correlation of the scales α and ω is demonstrated in figure 5. Error in the scale factors, which is a consequence of local measurement error, is indicated by bars. As is expected, error in conductivity scales is always greater than that in pressure head scales. A regression with the constraint of zero intercept, $\omega = B\alpha$, yields the following statistics: $B = 1.04 \pm 0.10$, standard error of estimate equals 0.284, and correlation coefficient $R = 0.97$. Since the eight scale factors cover the range of scale values for all locations having conductivity measurements, there is reason to expect that this regression represents the entire profile of the 1-m plots. When examining figure 5, remember that the bars represent the range of the errors not that of the standard deviations of the means. Thus, within the limitation of statistical error, the scale factors α and ω can be viewed as being equal.

The principle of scaling is valid if α and ω are identical over all locations. In actuality, statistical and computational errors will cause α and ω to differ. Thus, the principle of

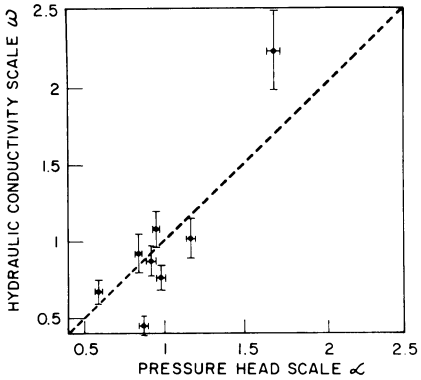

Fig. 5. Correlation of conductivity and pressure head scale factors for the 60 and 120 cm depths of the 1-m plots. Scale factors correspond to arithmetic mean values. Bars indicate standard errors.

TABLE 6.

REGRESSION COEFFICIENTS c FOR EQUATION (22). CORRELATION COEFFICIENT R AND STANDARD ERROR OF ESTIMATE σ. SAMPLE SIZE EQUALS 88 FOR EACH LOCATION IN THE 1-M PLOTS.

Depth	PLOT 1			2			3			4		
cm	C	R	σ	C	R	σ	C	R	σ	C	R	σ
15	1.00	1.00	0	1.00	1.00	0	1.00	1.00	0	1.00	1.00	0
30	0.32	0.64	0.019	0.75	0.94	0.005	0.71	0.91	0.006	0.82	0.96	0.006
45	0.61	0.92	0.011	0.59	0.95	0.006	0.62	0.93	0.006	0.72	0.96	0.007
60	0.86	0.83	0.017	0.62	0.96	0.006	0.66	0.96	0.005	0.74	0.96	0.007
75	0.79	0.79	0.018	0.69	0.96	0.007	0.66	0.97	0.005	0.79	0.92	0.010
90	0.97	0.90	0.012	0.80	0.98	0.006	0.66	0.95	0.006	0.97	0.94	0.009
105	0.90	0.78	0.017	0.86	0.98	0.005	0.80	0.96	0.006	1.00	0.95	0.008
120	0.89	0.76	0.017	0.85	0.93	0.010	0.66	0.97	0.005	0.97	0.97	0.006

scaling is considered to hold if α and ω represent estimates of the same expected mean for the scale distribution of each location. This distribution of scales within each location constitutes the local variation. Variation between locations of the estimated mean scales is the spatial variation.

Water content profiles and cumulative seepage

Storage is the instantaneous amount of water within the soil profile down to each soil depth and equals the integral of water content from the soil surface to each depth. The storage at each experimental sample time was calculated by the trapezoidal rule using the measured water content at 15 cm intervals to the 120 cm depth. Estimates of storage were obtained for each of the four neutron access pipes in each 1-m plot for 22 sample times from 0.6 to 56.6 days after cessation of steady state ponding conditions. The depth-averaged water content (2) was obtained by dividing the storage by the depth for each of the depths 15, 30, 45, 60, 75, 90, 105, and 120 cm. Cumulative seepage at each depth z is then equal to the difference in depth-averaged water content $\hat{\theta}_o - \hat{\theta}$, multiplied by depth z where $\hat{\theta}_o$ is the initial value.

Profiles of depth-averaged water content θ and water content $\hat{\theta}$ were found to have similar shapes during the drainage period. As a consequence, $\hat{\theta}$ and θ were found to satisfy a linear regression relation (22). Table 6 contains the regression coefficients c, standard error of estimate of depth-averaged water content, and correlation coefficient R for each depth in the 1-m plots. These regressions are based on the 88 individual measurements of θ and $\hat{\theta}$ at each soil location. Note that θ and $\hat{\theta}$ for the 15 cm depth are assumed equal. As indicated by table 6, $\hat{\theta}$ is usually less than θ at all locations because the intercept of the regression is approximately zero and c is less than 1. The standard error of c is about 0.03 for most locations.

The drainage curves for the depth-averaged water content as a function of time satisfy the following equation:

$$\hat{\theta} = \hat{\theta}_o - \frac{1}{\hat{\delta}} \ln[1 + J_o \hat{\delta} t/z]. \tag{164}$$

Estimated parameters for the least squares fit of (164) at each location are given in table 7. The fit is to the means of $\hat{\theta}$ over the four neutron meter measurements, and

is restricted to the first 31 days (19 points). Standard errors of estimate of $\hat{\theta}$ are within the limits of measurement error, indicating a good fit. Because (164) and the regression (22) imply that the draining profiles are described by equation (29) with δ equal to $c\hat{\delta}$, the flux is an exponential function of water content, given by equation (28). It must be emphasized that this conclusion is unique to the particular boundary conditions and the validity of the regression relation (22). Table 7 shows that the initial flux J_0 increases with depth. Flux as an explicit function of time is given by equation (30) and the parameters of table 7. According to equation (164), after a sufficient period of drainage, the spatial variability of flux is primarily due to variation in $\hat{\delta}$. Five to ten days appears to be such a period in the case of the 1-m plots.

The exponential flux model (28) can also be verified directly by linear regression for the logarithm of finite difference estimates of flux and the water content. Table 8 contains the estimates for the parameters of equation (28) obtained from such regressions.

Indirect scaling of flux

The flux described by the exponential model (28) can be scaled indirectly if the drainage curves given by equation (29) are similar over the soil profile, and the drainage curves are similar if equation (29) with a common value of δ can be least squares fit to the measured water content profiles. Simultaneous fit of equation (29) using a common δ for all locations was achieved with the drainage scaling program of appendix D. Scale factors ω and parameters for the scaled flux are presented in table 9. Observe that according to the definition (59), the actual variable being scaled, that is, reduced, is the time. Measured drainage curves for all 32 locations in the 1-m plots are shown in

TABLE 7.

PARAMETERS FOR THE DRAINAGE EQUATION (164), BASED ON 31 DAYS OF DRAINAGE IN THE 1-M PLOTS. SAMPLE SIZE = 19 AT EACH LOCATION. σ DENOTES THE STANDARD ERROR OF ESTIMATE OF WATER CONTENT.

Depth	PLOT 1				PLOT 2			
cm	$\hat{\theta}_0$	J_0	$\hat{\delta}$	σ	$\hat{\theta}_0$	J_0	$\hat{\delta}$	σ
15	0.393	0.24	40.2	0.007	0.394	0.27	90.3	0.005
30	0.390	0.79	48.5	0.005	0.393	0.65	79.5	0.005
45	0.401	2.53	47.7	0.005	0.395	1.67	67.4	0.005
60	0.408	4.87	46.7	0.006	0.399	3.72	59.8	0.005
75	0.396	6.94	52.0	0.005	0.401	6.28	56.3	0.006
90	0.399	8.95	52.0	0.006	0.403	9.04	54.5	0.007
105	0.401	10.47	52.7	0.006	0.404	11.16	53.4	0.007
120	0.403	11.17	53.5	0.006	0.406	14.15	53.4	0.007
	PLOT 3				PLOT 4			
15	0.380	0.27	82.5	—	0.406	1.37	66.6	0.004
30	0.383	0.56	61.0	0.018	0.404	3.07	64.2	0.004
45	0.388	0.55	69.8	0.004	0.402	5.11	58.6	0.004
60	0.393	1.36	69.5	0.004	0.399	6.90	53.5	0.004
75	0.398	2.83	67.5	0.005	0.396	8.20	51.1	0.005
90	0.403	5.12	65.5	0.005	0.395	9.33	50.9	0.005
105	0.406	6.68	64.0	0.005	0.394	9.66	51.3	0.006
120	0.409	7.75	60.6	0.005	0.396	11.20	51.7	0.006

TABLE 8.

REGRESSIONS FOR FLUX EQUATION (28) FOR 20 LOCATIONS IN THE 1 METER PLOTS. θ_0 AND SAMPLE SIZE HAVE THE VALUES IN TABLE 1. σ DENOTES ERROR FOR δ AND STANDARD ERROR OF ESTIMATE FOR $\ln J$. CORRELATION COEFFICIENT R. J_0 (CM/DAY) IS THE GEOMETRIC MEAN.

Depth	PLOT 1					PLOT 2				
cm	J_0	δ	σ	$\sigma[\ln J]$	R	J_0	δ	σ	$\sigma[\ln J]$	R
60	2.7	30.4	5.7	0.741	0.82	1.9	26.1	6.2	0.816	0.77
75	3.4	35.5	6.1	0.715	0.84	2.9	27.5	5.8	0.783	0.81
90	4.1	41.3	7.0	0.750	0.85	3.6	31.4	6.2	0.764	0.81
105	3.7	51.7	8.7	0.832	0.84	4.3	34.1	6.4	0.764	0.83
120	4.9	51.2	10.3	0.856	0.80	5.9	36.0	6.7	0.785	0.83
			PLOT 3					PLOT 4		
60	1.0	31.6	11.1	1.058	0.61	2.3	29.6	5.7	0.722	0.82
75	2.1	32.7	8.9	0.941	0.70	2.8	33.1	6.1	0.718	0.83
90	3.4	36.1	7.1	0.823	0.81	3.4	42.1	9.1	0.859	0.79
105	3.1	38.5	8.5	0.843	0.77	3.3	40.6	9.9	0.944	0.75
120	3.5	32.5	5.8	0.762	0.82	4.7	38.9	9.8	0.991	0.74

figure 6. The water content data shown are for 31 days of drainage and consist of 608 measurements of average values. Figure 7 demonstrates the scaling of the data in figure 6. Water content measurements were coalesced in figure 7 by plotting them in terms of reduced time (59) for each location, relative to a common initial value $\bar{\theta}_0$. Ideally, if the measured drainage curves were exactly similar, then the θ in terms of reduced time would coalesce into the drainage curve given by (60), which is indicated in figure 7. The measured θ deviate from that curve with a pooled standard deviation equal to 0.008; a value comparable to the measurement error.

Examples of the scaled drainage curves for the 120 cm depth are shown in figure 8. These curves expressed in terms of depth-averaged water content are compared with measured values, and were obtained by multiplying calculated θ by c. A corrected c based on regression of mean differences $\hat{\theta}_0 - \hat{\theta}$ and $\theta_0 - \theta$, however, is used instead of c from table 6. This c is calculated as part of the scaling program of appendix D. In figure 8, drainage scaled in terms of θ is converted to $\hat{\theta}$ so that cumulative seepage and flux can be estimated directly. A drainage equation in terms of $\hat{\theta}$ with a common $\hat{\delta}$ for all locations could also be scaled directly; however, the advantage of scaling in terms of θ is that an estimate of scaled conductivity as a function of θ for unit hydraulic gradient conditions is obtained.

The average and standard deviation of J_0 over all 32 locations equal 6.86 cm/day and 5.06 cm/day, respectively, computed with the scale relation (56) and the scale factors of table 9. It should be noted that these J_0 which are estimated relative to a common δ differ from those of table 7, for which δ depends on the location. Statistical variations allow both descriptions of the drainage profiles to be valid. Of course, the parameters of table 7 provide the best possible fit of equation (164). In the ideal case, if the similarity of the measured drainage curves were exact, the δ (or $\hat{\delta}$) obtained from independent fitting for each location would be constant. Such an ideal situation, however, could never be achieved with field measurements, in view of the inherent local variability.

Thus, within the limitations of demonstrated similarity, the spatial variability of the drainage flux is represented by the scale factors of table 9, where the variability is relative to identical initial water contents (i.e., a common reference value $\bar{\theta}_o$). It will now be demonstrated that the conductivity manifests this same spatial variability over the soil profile.

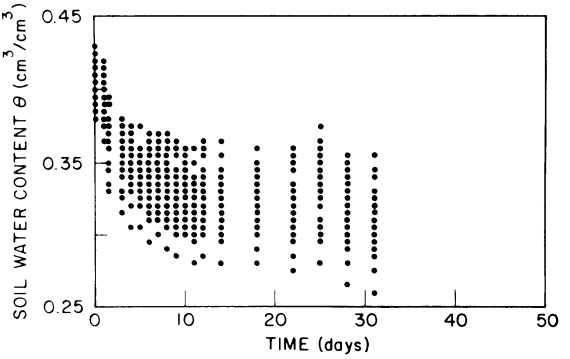

Fig. 6. Soil water content versus time for 19 measurements at each of 32 locations in the 1-m plots: Depths 15, 30, 45, 60, 75, 90, 105, and 120 cm in four plots. Sample size = 608.

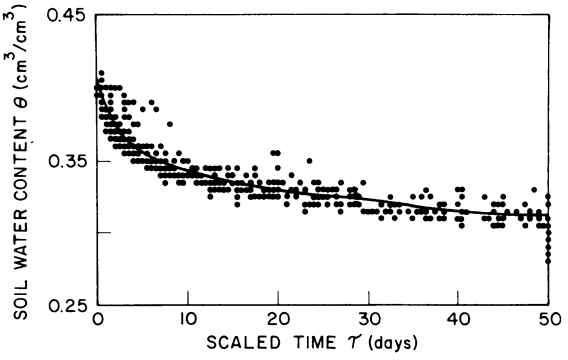

Fig. 7. Scaled drainage curves for the data of figure 6. Soil water content versus reduced time τ, equation (59). Curve is equation (60) with $\bar{\theta}_o$ = 0.408, δ = 50, J_m = 5.29 cm/day, and z_m = 120 cm.

TABLE 9.

SCALE FACTORS ω AND PARAMETERS FOR THE SCALED EXPONENTIAL FLUX EQUATION (28) FOR 32 LOCATIONS IN THE 1-METER PLOTS. θ_o HAVE THE VALUES IN TABLE 1, AND c DENOTES THE CORRECTED REGRESSION COEFFICIENT. COMMON PARAMETERS: δ = 50.0, $\bar{\theta}_o$ = 0.408, and J_m = 5.29 CM/DAY.

Depth	PLOT 1			2			3			4		
cm	J_o	ω	c	J_o	ω	c	J_o	ω	c	J_o	ω	c
15	0.35	0.375	1.00	0.10	0.195	1.00	0.067	0.190	1.11	0.62	0.361	0.99
30	3.45	0.696	0.60	0.53	0.483	0.69	0.29	0.332	0.75	2.40	0.822	0.84
45	12.35	0.839	0.65	4.65	0.937	0.53	1.03	0.487	0.62	7.37	1.555	0.77
60	12.5	0.084	0.81	11.6	1.377	0.60	2.54	0.596	0.56	9.74	2.599	0.85
75	9.68	1.136	0.87	16.4	1.592	0.68	6.27	0.787	0.57	7.52	2.120	1.0
90	8.07	1.146	1.00	14.6	1.583	0.80	8.68	0.797	0.64	5.39	1.544	1.19
105	4.82	0.932	1.26	14.3	1.415	0.87	5.85	0.638	0.82	5.49	1.445	1.20
120	5.45	0.648	1.22	16.9	1.325	0.89	11.2	0.859	0.73	9.19	1.107	1.03

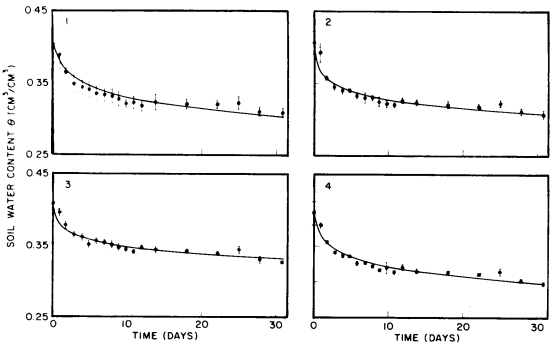

Fig. 8. Depth-averaged soil water content versus time for 120 cm depth for the four 1-m plots. Curves are given by equation (164) with parameters from table 9, based on scaled flux. Circles indicate means of four measurements and bars indicate standard deviation.

Comparison of scale factors

Correlation of the conductivity scale factors in table 4 and the drainage scale factors in table 9 for the 60, 75, 90, 105, and 120 cm depths is shown in figure 9. The drainage scale factors were renormalized for those depths, so that both scale factor distributions are compatible with respect to representation in terms of θ. A regression of conductivity on drainage scale factors with zero intercept has the following statistics: slope equal 0.998 ± 0.024, standard error of estimate equal 0.115, and correlation coefficient R equal 0.995. Thus the scale factors can be considered as equal, for all practical purposes. Therefore the relative conductivity over the soil profiles is essentially described by the drainage scale factors, at least for 60 to 120 cm. Moreover, these results tend to verify the new scale relations for flux and reduced time, (iv) and (v). Figure 10 shows graphs of the drainage scale factor profiles for the four plots.

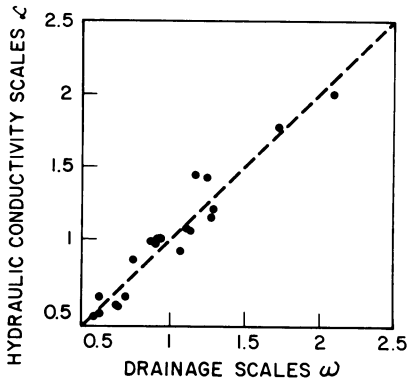

Fig. 9. Hydraulic conductivity scales α versus drainage scales ω for the 60, 75, 90, 105, and 120 cm depth of the 1-m plots. Scales are computed relative to the scaling relations (38) and (56) in terms of water content with β = 31.6, K_m = 6.16 cm/day, δ = 50.0, J_m = 9.39 cm/day, and $\bar{\theta}_0$ = 0.411.

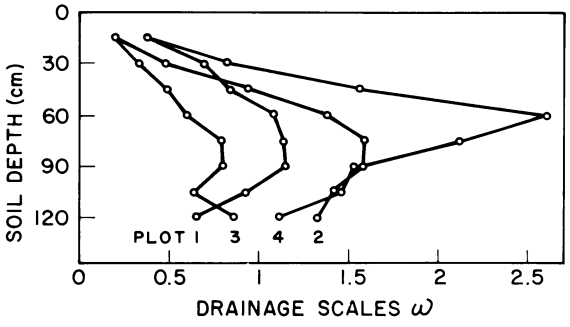

Fig. 10. Drainage scale factor profiles for the 1-m plots. Scales ω are from table 9.

The drainage scale factor profiles in figure 10 show a repeated pattern: they are minimum at the profile ends and maximum near the middle. Scale factors α obtained for the pressure head show a similar pattern for the entire experimental field plots (figure 8 of Part I), which seems to suggest a layer effect. This conclusion is drawn from the correlation of ω and α. Inasmuch as the pressure head and conductivity scale factors are correlated, assuming a transitive relationship, correlation of conductivity and drainage scale factors implies the result. Of course, the suggested correlation of ω and α can be tested directly by scaling flux in terms of degree saturation s.

Equality of conductivity and drainage scale factors provides an immediate application. Apparently it is sufficient to measure conductivity at only one soil location in a set of similar soil locations. Conductivity for all other locations can then be generated by using the scale relations and the determined drainage scale factors. Similarity for a particular soil region is first established by examination of the drainage profile scaling. Such a method, which is based mainly on measurement of water content, substantially reduces the need for tensiometer installation.

Hydraulic conductivity: corrected estimates

Two methods for calculating a corrected conductivity which include the effect of hydraulic gradient are proposed. Both methods use improved estimation of the pressure head gradient based on drainage scale factors. Method 1 which is based on the gradient of the scale factors is given by equations (76) and (77), and method 2 which is based on an integral equation for hydraulic head difference is given by equations (79) and (81).

Figure 11 compares conductivity estimated by those methods with the finite difference estimates based on pressure head gradient between 60 and 120 cm (table 1). The comparison is restricted to the 60 and 120 cm depths where pressure head was measured. It should be noted that the finite difference estimates used to determine the regressions in table 1 consistently underestimate conductivity, because conductivity over each measured interval of water content is associated with the higher end of those intervals (i.e., an advanced time estimate of conductivity, equation (50)), rather than with the midpoint. This causes a corresponding underestimate of K_0 for each location. Translation of the regression (32) by a value of θ equal to half the interval θ_1 to θ_0, where θ_1 is the first measurement following θ_0, would approximately correct this discrepancy, i.e., assign the value K_0 to $(\theta_0 + \theta_1)/2$ instead of θ_0, so that the corrected steady state conductivity equals $K_0 \exp(\beta(\theta_0 - \theta_1)/2)$. Corrected values average about 40 percent higher than original K_0. Moreover, K_0 is underestimated, because the values in table 1 are geometric means instead of arithmetic means. In any event, the regressions of table 1 best represent the approximate graph of conductivity on semi-log coordinates. For reasons similar to those mentioned, steady state flux J_0 estimated indirectly from the drainage equation (table 7) is consistently greater than the corresponding finite difference estimate (table 8). The result is a greater initial conductivity for both correction methods. On the other hand, the minimum conductivity for the experimental range of water content is usually smaller for the correction methods. This is partially a consequence of the fact that minimum flux predicted by table 9 is usually less than that of table 8. It is the relative value of δ that primarily determined the minimum value of flux over the measured range of water content.

The conductivity of method 1 is calculated using the parameters of scaled pressure head (Part I, table 1) and flux (table 9) and an average gradient $d\alpha/dz$ equal to -0.0055, while the gradient of θ_0 is based on the measured finite difference estimates.

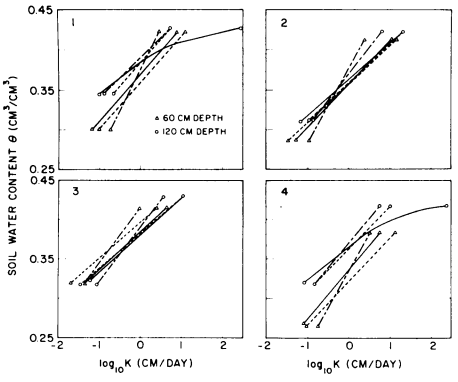

Fig. 11. Hydraulic conducitivity for the 60 and 120 cm depths of the four 1-m plots estimated by two analytical methods. Method (1): solid line is based on equations (76) and (77) with table 1 of Part I and table 9. Method (2): dash line is based on equations (79) and (81) with table 11. Measured conductivity of table 1 is indicated by a broken line.

TABLE 10.

PARAMETERS FOR LEAST SQUARES FIT OF INTEGRAL EQUATION (79). COMMON REFERENCE $\bar{\theta}_o$ EQUALS 0.411.

Plot	A	σ[A]	B	σ[B]	σ[h]	D.F.
1	0.896	0.031	10.22	0.61	2.7	17
2	0.762	0.043	1.81	0.65	3.6	17
3	1.051	0.076	−1.18	1.09	7.8	17
4	0.931	0.106	7.62	1.48	6.0	17
Pooled estimates:						
	0.973	0.094	4.78	1.34	16.3	74

TABLE 11.

STEADY STATE CONDUCTIVITY K_o FOR THE 60 AND 120 CM DEPTHS IN THE ONE METER PLOTS. K_o AND β ARE GIVEN BY EQUATION (81) AND TABLE 10.

Depth cm	PLOT 1	2	3	4
60	12.5	15.3	2.4	13.0
120	5.2	21.8	10.8	9.6
β	39.8	48.2	51.2	42.4
Pooled estimates:				
60	12.2	12.0	2.6	11.5
120	5.2	16.6	10.5	9.3
β	45.2	45.2	45.2	45.2

$K_m = 5.52$ cm/day, $\bar{\theta}_o = 0.408$

Method 1 gave extreme initial values at 120 cm in plots 1 and 4, but was consistent with the condition that pressure head gradient approaches zero at deeper depths. Conductivity by method 2 is calculated using the parameters of table 11, which are obtained from the estimates of A and B for individual plots given in table 10. The parameters A and B were obtained by least squares fit of (79) to measured water content and pressure head. In table 10, pooled estimates represent the combined four plots. The values of B in table 10 indicate that drainage in plot 3 is distinctly different than that of the other plots: pressure head is positive, whereas it is otherwise negative. Indeed, substitution of the parameters of tables 1 and 8 for plot 3 into equation (82) yields a negative hydraulic gradient, consistent with actual measurement. Similar substitutions for plots 1, 2, and 4 yield positive gradients. Therefore the pooled estimates of A and B associated with scaling the entire 1-m plots data includes an erroneous implication that negative hydraulic gradient conditions prevail throughout. This error is reflected by the increased standard error of estimate of hydraulic head difference, when the data are pooled. Evidently, plot 3 is not compatible with the scaling of conductivity by this method. Figure 12 compares the conductivity scaled by method 2, using the pooled estimates of K_0 and β. Even though the pressure head gradient is incorrectly predicted by a scaled exponential flux, the scaled exponential conductivity still seems to provide a reasonable approximation at the 60 and 120 cm depths.

In general, scaling of conductivity for the case of pressure head gradient having a fluctuating sign requires the application of equation (70) with a δ that is variable. Even then, scaling in terms of simple exponential models for flux and conductivity may not yield correct results. In this situation, more general model functions would be required to properly predict the pressure head gradient.

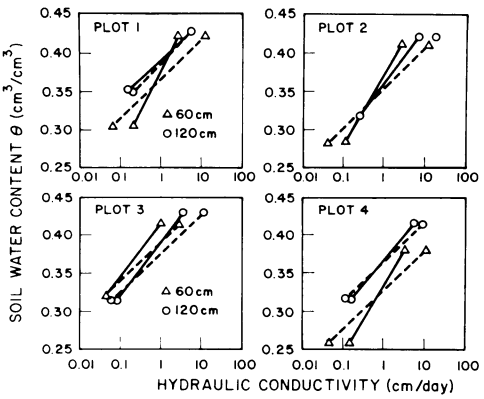

Fig. 12. Scaled conductivity for the correction method based on the integral equation (79) for the 60 and 120 cm depths of the four 1-m plots. The dash line indicates conductivity for the pooled estimates of K_0 and β in table 11, and the solid line indicates conductivity of table 1.

Millington-Quirk conductivity

An example calculation of Millington-Quirk conductivity (90) for the 60 cm depth in plot 1 is shown in figure 13. Using the program of appendix E, the conductivity (95) was matched with the parameters of table 3, for θ_{min} equal to 0.3, and the soil-water characteristic was determined by equation (94) with the parameters of table 1 in Part I. Here the conductivity (90) is based on 20 pore classes. Larger numbers of pore classes yield approximately the same conductivity for water contents below θ_0; however, for θ greater than θ_0 the estimates that approach saturated conductivity become greater. Upper and lower limits of θ corresponding to physical pore size limits were assumed to

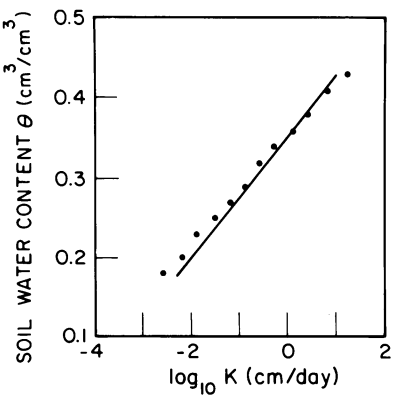

Fig. 13. Millington-Quirk conductivity for the 60 cm depth in plot 1 of the 1-m plots. Dots indicate equation (90) for 20 pore classes with matching factor equal to 0.013 and soil-water characteristic (94). Solid line indicates the conductivity equation (95) with parameters of table 3. Parameters for equation (96) are K_o = 8.21 cm/day, θ_o = .422, β_1 = 30.5, and β_2 = −11.1. Matching is for θ_{min} = 0.3.

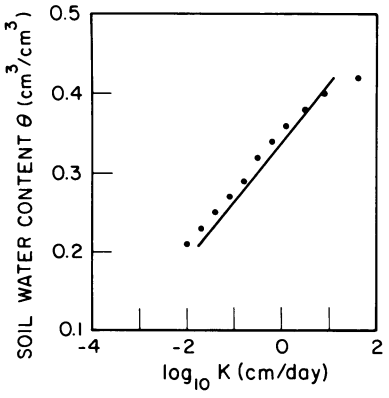

Fig. 14. Millington-Quirk estimate of scale mean conductivity for 72 locations in the experimental field. Dots indicate equation (99) for 20 pore classes with s = θ/θ_o where θ_o = 0.42 and matching factor equals 0.041, for θ_{min} = 0.3. Soil-water characteristic (94) is determined by a_m = −114 cm and ϱ = −11.74. Solid line indicates conductivity equation (95) with K_o = 12.7 cm/day, β = 31.0, θ_o = 0.42. Parameters for equation (96) are K_o = 12.7, θ_o = 0.42, β_1 = 35.7, and β_2 = 7.2.

equal 0.43 and 0.1, respectively. This upper limit of θ corresponds to a −12 cm pressure head. The estimated matching factor is 0.013 with a standard error of 0.001. Although equations (90) and (95) are matched only in the θ interval 0.3 to 0.422, their graphs continue to agree below this interval, and substantial departure does not occur until θ approaches 0.1. Least squares fit of the extended equation (96) with the restriction that K_o = 8.21 cm/day yields β_1 = 30.5 and β_2 = −11.1, and indicates that the logarithm of Millington-Quirk conductivity is approximately linear, except near saturated water content.

Figure 14 shows the Millington-Quirk calculation of scale mean conductivity for the plots of the experimental field. The conductivity (99) was calculated in terms of an approximate saturation variable θ/θ_o, using the experimental field's scaled soil-water characteristic curve, which is determined by a_m = −114 and b = −4.93, and by using the scale mean conductivity for the 1-m plots, matched with equation (160) to the 72 experimental field scale factors. The scale mean K_m equals 12.7 cm/day and was obtained by matching scale factors for the 60 and 120 cm depths in the 1-m plots and experimental field. In figure 14, the conductivity is presented in terms of θ for θ_o equal 0.42. Matching of Millington-Quirk conductivity in the approximate saturation interval 0.71 to 1.0 gave a matching factor equal to 0.041 with a standard error of 0.015.

Again Millington-Quirk conductivity was based arbitrarily on 20 pore classes. Increasing the number of pore classes tends only to increase conductivity approaching saturation. Least squares fit of the extended model (96) with the restriction that K_m equals 12.7 cm/day and θ_o equals 0.42 yields β_1 = 35.7 and β_2 = 7.2 with correlation coeffi-

cient of 0.98. The fit of (96) for 50 pore classes yields $\beta_1 = 33.9$ and $\beta_2 = 0.5$ with correlation coefficient of 0.99. Observe that the Millington-Quirk estimate of scale mean conductivity for the experimental field plots is thus based on two matching procedures: (1) the conductivity measured in the 1-m plots is matched to the experimental field scale factor distribution—a spatial extension of conductivity, and (2) the Millington-Quirk conductivity is matched to the exponential model of the scale mean—an extension of the conductivity function (96) to lower water content values. In figure 14, the Millington-Quirk conductivity is compared with the exponential model (95) when extended beyond the range of measurement. For the most part, the logarithm of Millington-Quirk conductivity is approximately linear, but it is slightly less than that predicted by the exponential model. This result, which also appears in figure 13, is qualitatively consistent with the relationship of the exponential model and measured conductivity seen in figures 2 and 3.

Predictions of the conductivity for each location in the experimental field are obtainable from the scale relation (100). The error in this Millington-Quirk method of estimating conductivity is then dependent on the errors in K_m, the matching factor K_s/K_{sc}, and the scale factor α. Apparently, the variance of ln K(s) propagates as a linear combination of the variances of the logarithms of K_m, K_s/K_{sc} ($= K_{act}/K_{calc}$), and α, neglecting variations in other relevant parameters.

Results and Discussion: Flux

Variability of cumulative seepage and flux

The implications of spatial variability of the hydraulic conductivity are demonstrated in figures 15 and 16 for the 120 cm depth in the 1-m plots. Figures 15 and 16 depict the depth-averaged water content and flux under unit hydraulic gradient conditions for the two extreme values of steady state conductivity occurring in plots 1 and 2. In those figures, the depth-averaged water content, $\hat{\theta}$, defined by equation (27) and the flux, which equals $-z d\hat{\theta}/dt$ for z positive downward, are based on the parameters of table 2 for the scaled exponential conductivity. For the sake of comparison, however, the $\hat{\theta}$ curves are plotted with the same initial value $\hat{\theta}_o$, without altering estimates of

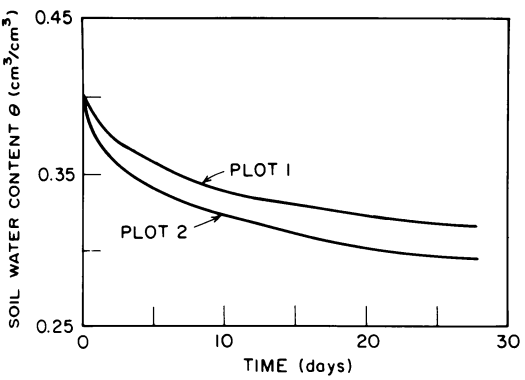

Fig. 15. Depth-averaged water content at the 120 cm depth versus time for plots 1 and 2 of the 1-m plots. Curves are given by equation (27) with parameters of table 2.

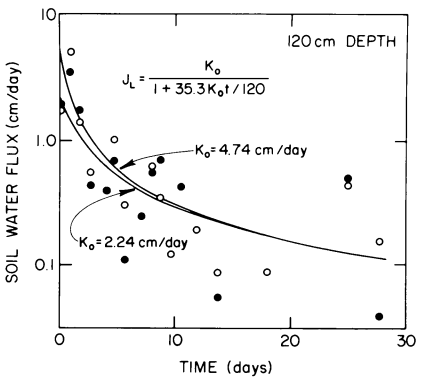

Fig. 16. Flux at the 120 cm depth versus time for plots 1 and 2 of the 1-m plots. Curves are given by equation (3) for figure 15. Circles indicate measured flux, based on finite difference estimates.

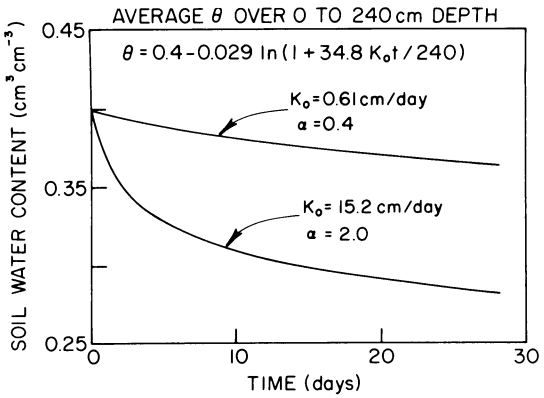

Fig. 17. Depth-averaged water content for the 240 cm depth of the experimental field versus time for two values of scale factor α. Hydraulic conductivity parameters correspond to those given in table 2.

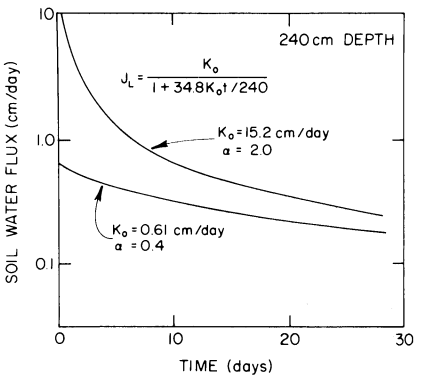

Fig. 18. Soil water flux at the 240 cm depth of the experimental field versus time corresponding to figure 17.

seepage and flux. Then the difference of cumulative seepage between plots at any time simply equals the difference in $\hat{\theta}$ multiplied by the depth, 120 cm. Thus corresponding to the scale factors 0.769 and 1.118 the predicted difference in cumulative seepage after 25 days is 2.4 cm. The mean flux compared with actual finite difference estimates in figure 16 manifests an asymptotic convergence property characteristic of scaled conductivity: after 10 days of drainage the flux within each plot is approximately the same. Figures 17 and 18 depict the simulation of cumulative seepage and flux for scale factors that are typical of the entire experimental field plots. The scale factors are 0.4 and 2.0, and the parameters used in figures 15 and 16 are assumed for comparison. At the 240 cm depth, the difference in cumulative seepage is 14.4 cm in 5 days and 20.4 cm in 25 days. These figures also show the asymptotic convergence of flux, but to a less extent. As indicated by equation (137), the residual difference in cumulative seepage after a sufficient drainage period is proportional to the difference in the logarithms of α, and the variability in $\ln \alpha$ is a direct indication of variability in cumulative seepage.

Approximate drainage curves

At deep depths the condition of unit hydraulic gradient is often found to be a reasonable assumption, and the drainage curve for the depth averaged water content (storage divided by depth) can be described approximately by equation (27), when the conductivity satisfies a simple exponential equation (25). Figure 19 demonstrates the prediction of depth-averaged water content based on the scaled conductivity for the 120 cm depth in the 1-m plots. In figure 19, the deterministic drainage curve (B) obtained by substitution of a mean K_0 into equation (27) and the local stochastic average curve (C) obtained from equation (111) are compared with the directly fit drainage curve (A) of equation (164). Curve (A) is based on table 7, and curves (B) and (C) are based on the scaled arithmetic mean conductivity and standard deviations of table 3. For the curves (B) and (C), $\hat{\beta}$ equals $b/\theta_0 c$, and the values are 34.3, 36.4, 45.9, and 32.3 for plots 1 through 4, respectively. The best possible fit of equation (27) is represented by that of curve (A). For all locations, the deterministic curve eventually gives values lower than those measured, whereas the average curve, which is adjusted to account for local variation of K_0, yields higher values. Evidently, the correction to the average curve (C), which depends on the local variance of K_0, is over estimated. This can be a consequence

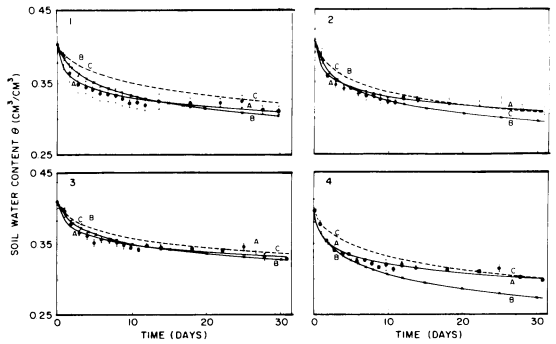

Fig. 19. Drainage curves for the depth-averaged water content at the 120 cm depth in the four 1-m plots. Curve (A) is given by equation (164) with table 7. Curves (B) and (C) are the deterministic and local average forms of equation (27), respectively, based on table 3. Bars indicate standard deviation of measurements.

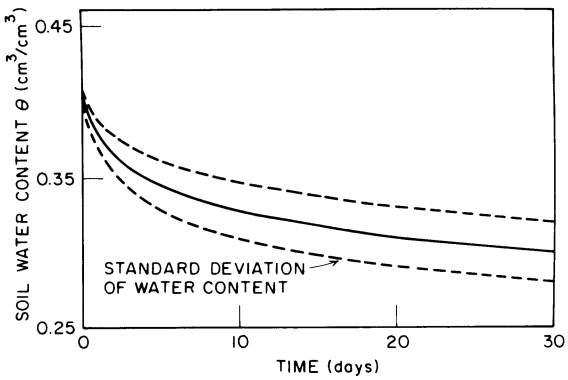

Fig. 20. Spatial average and standard deviation of depth-averaged water content for the 120 cm depth of a 1-m plots.

of equating the entire standard error of estimate to the standard deviation of K_o. Nevertheless, after 25 days of drainage, curve (C) provides good predictions of the cumulative seepage. The cumulative seepages at 30 days based on the (A) curves are 11.2, 11.8, 9.5, and 11.5 cm for the four plots. Such verification of the prediction of local seepage given by equation (27) is a necessary preliminary to estimation of the spatial average of cumulative seepage.

Figure 20 shows the spatial average and standard deviation of the depth-averaged water content at 120 cm in the 1-m plots. That is, figure 20 represents the average over plots of the deterministic drainage curves (B) of figure 19. This average was calculated with equation (122), neglecting local variances of K_o, while considering only the effect of the spatial variance v_o^2. Thus, the spatial average is obtained by subtracting the average difference $\bar{\Delta}$ from a reference value of average $\hat{\theta}_o$. Standard deviation indicated by dash curves is that for Δ given by (125), so that this does not include the variation of θ_o. Indeed, the indicated standard deviation is that associated with spatial variation of cumulative seepage, not with that of storage. The average parameters used in figure 20 are $\overline{K}_o$ = 5.08 cm/day, $\hat{\beta}$ = 37.2, and $\hat{\theta}_o$ = .404, while the spatial standard deviation of K_o is v_o = 1.66. Now since the average local standard deviation of K_o is 5.7 and is greater than v_o, the major contribution to the total variance (126) required in equation (122) is from the local variability of K_o, which was neglected in figure 20. For the same reason, the indicated standard deviation is mainly due to local variability of K_o. Thus for this particular range of scale values, spatial variability observed at the 120 cm depth can be viewed as actually a consequence of local measurement errors. Similar results and conclusions hold for the average $\hat{\theta}$ which is simulated with the K_o associated with the experimental field's scale factors for 120 cm, table 12. On the other hand, simulation of $\hat{\theta}$ for the 240 cm depth in the experimental field shows mainly spatial variability.

TABLE 12.

MEAN AND STANDARD DEVIATION OF K_o (CM/DAY) SIMULATED WITH THE PRESSURE HEAD SCALE FACTORS FOR PLOTS OF THE EXPERIMENTAL FIELD.

Depth	60 cm		120 cm		240 cm	
PLOT	K_o	σ	K_o	σ	K_o	σ
1	2.9	3.9	5.6	4.6	2.7	2.2
2	5.6	4.4	3.9	1.7	6.3	2.8
3	2.5	1.3	1.9	1.2	45.6	40.8
4	6.3	6.1	2.0	1.5	798.4	591.0
5	4.8	5.7	3.1	1.8	2.3	1.3
6	3.5	2.1	9.9	6.1	2.6	2.0
7	2.6	1.6	4.8	4.0	24.4	19.4
8	4.2	2.2	3.6	2.1	515.8	191.0
9	2.4	1.7	6.0	3.2	3.6	2.0
10	11.3	5.8	5.6	2.7	14.8	9.7
11	10.5	11.4	7.4	8.6	74.1	45.2
12	20.3	9.4	6.9	5.1	86.7	39.1
Avg.	6.4	4.6	5.1	3.5	131.4	78.9
std. dev.	5.3		2.3		254.4	
K_m	11.1		15.6		15.6	
β	45.1		37.2		37.2	

Simulation of experimental field flux

The spatial extension method associated with equation (160) was applied to the 12 plots of the experimental field, and drainage was simulated with the pressure head scale factors for the 120 and 240 cm depths, by using equation (127) for unit hydraulic gradient conditions. The matched conductivity measurements were obtained from the scaled conductivity at 60 and 120 cm in the 1-m plots. Cumulative seepage predicted by this method represents the maximum possible drainage that could occur for a saturated soil profile with spatial variability typical of the experimental field, when there is no evaporation or transpiration. Actual seepage at deep depths could be estimated by subtracting total water lost through evapotranspiration.

Table 12 contains the mean steady state conductivity and standard deviation for the 60, 120, and 240 cm depths in the experimental field. An estimate of K_m and β restricted to each depth is also provided, where β is an average obtained from the 1-m plots. Standard deviation of K_o was estimated from that of the local distribution of $\ln \alpha$, by assuming a log-normal distribution. At the 120 cm depth, the mean and standard deviation of K_o from table 12 are 5.07 cm/day and 2.33 cm/day, respectively. The graph of cumulative seepage at this depth in the experimental field is nearly identical to that of figure 20, since the statistics are essentially the same. This result assumes that β is constant over locations. Of course, variation of β could invalidate the result. However, a common β is implied by the principle of soil similarity.

Simulated average $\hat{\theta}$ for the 240 cm depth in the experimental field is shown in figure 21, which is based on table 12. The corrected average $\hat{\theta}$, which is based on equation (122), is substantially greater than the corresponding deterministic estimate based on the average K_o equal to 131 cm/day (dashed curve). Drainage curves for two other

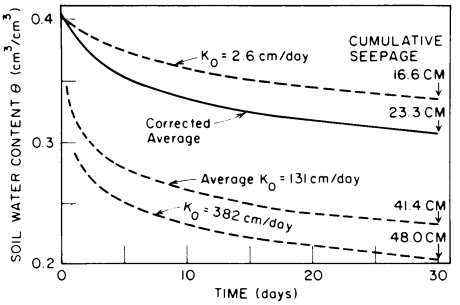

Fig. 21. Spatial average of depth-averaged water content for 240 cm simulated with scale factors for the experimental field. Solid curve is the corrected average with parameters of table 12. Dashed curve indicates deterministic estimate for the average K_0 equal 131 cm/day.

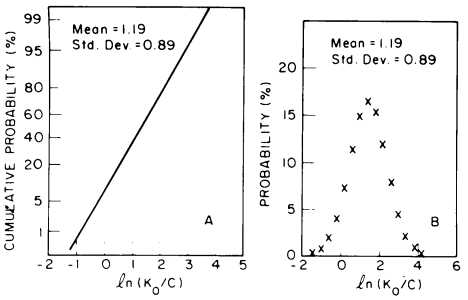

Fig. 22. a. Cumulative probability of the pooled distribution for 1n (K_0/c) at the 120 cm depth in the 1-m plots.

b. Pooled probability distribution of 1n (K_0/c), equation (153) with K_0 = 4.9 (σ = 5.4) and c = 0.85 (σ = 0.15). The class interval length equals 0.4.

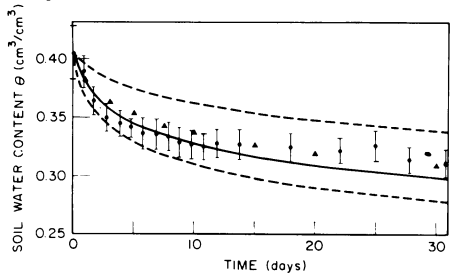

Fig. 23. Mean depth-averaged water content for the 120 cm depth in the 1-m plots. Triangles denote the expected mean based on figure 22, and dashed lines denote the expected standard deviation. Solid curve indicates the deterministic estimate based on averages β = 30.8, K_0 = 4.9 cm/day, θ_0 = 0.404, and c = 0.85. Circles indicate measured values and bars the standard deviation.

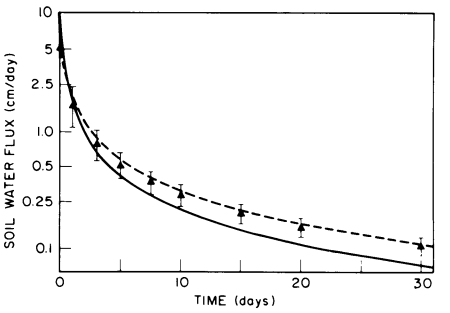

Fig. 24. Mean flux for the 120 cm depth in the 1-m plots. Expected mean and standard deviation are indicated by triangles and bars. The deterministic estimate corresponding to figure 23 is indicated by a dashed curve. Measured average based on table 7 is indicated by a solid curve.

extreme values of K_0, 2.6 and 382 cm/day, are also indicated in figure 21. At this depth, the main component of variability is spatial, with a standard deviation of K_0 equal to 254 cm/day. The initial value of $\hat{\theta}_0$ is that used in figure 20. But because cumulative seepage and flux do not depend on initial depth-averaged water content, this value is arbitrary. Simulated flux is simply the negative slope of the drainage curve multiplied by depth.

Distributions of water content and flux

If the local variability of K_0 is described by a log-normal distribution, the statistics of cumulative seepage and flux can be estimated with the pooled distribution of 1n (K_0/c) given by equation (153). Moreover, the distributions of random functions of K_0 and c can be obtained.

Figure 22 is the pooled probability distribution of 1n (K_0/c) for 120 cm in the 1-m plots, and is based on table 2. This distribution, which is the sum of four normal distributions, describes both the local and spatial variability. The graph of cumulative probability for figure 22, which is nearly linear, indicates that the pooled distribution is again approximately normal, with mean and standard deviation of 1n (K_0/c) equal 1.19 and 0.89, respectively. In figures 23 and 24 the average and standard deviation of

depth-averaged water content and flux corresponding to figure 22 are shown for some selected drainage times. The measured flux, which is indicated by a solid curve in figure 24, is the mean of the best fit curves (A) of figure 19. A dashed curve indicates the deterministic flux given by equation (132) with mean values of β, K_o, and c equal to 30.8, 4.9 cm/day, and 0.85, respectively. In this case, the deterministic curve approximates the mean flux computed from the sample distribution, figure 22. The experimental flux, however, is less than both estimates based on the unit hydraulic gradient assumption. This would be the expected relationship, since the hydraulic gradient was usually negative. Expected values of depth-averaged water content (assuming that θ_o equals 0.404 without error) and measured values are compared in figure 23. In that figure, the solid curve represents the deterministic estimate given by equation (27) with the mean values of β, K_o, and c. Both the flux and $\hat{\theta}$ are computed with a common value of β, so that scaling holds. As seen in figure 23, the expected standard deviation envelopes both measured and deterministic values of $\hat{\theta}$. For the most part, measured standard deviations also fall within that envelope. Consistent with the qualitative behavior predicted by equation (127), the expected $\hat{\theta}$ is greater than the deterministic estimate, since expected seepage is reduced by the variance of K_o. Moreover, the deterministic cumulative seepage is greater than that actually measured after 10 days, which is consistent with the greater estimate of deterministic flux.

Figures 25 and 26 show the cumulative probability of the difference $c\Delta$, equation (139) multiplied by c, and the flux at some selected times. These distributions correspond to figure 22. By 3 days of drainage the distribution of $c\Delta$ is essentially normal, reflecting the normal distribution of $\ln (K_o/c)$. Initially, the flux, being equal to K_o, is log-normally distributed; but after three days, the skewness is diminished; and the distribution's tail is cut off by the asymptotic property of equation (132). By 30 days the flux distribution has nearly reduced to a single value, as a consequence of soil similarity.

In general, the pooled distribution of K_o for local and spatial variability is given by a weighted sum of log-normal distributions associated with each location. The distribution for each location represents local (within sample) variation, while the weighted sum of these (pooled) represents spatial (between samples) variation. This pooled distribution for a particular set of soil locations may or may not again be log-normal, depending on soil similarity of the set of locations. Indeed, a pooled distribution can be multimodal, and thus not be a simple log-normal distribution. In the case of the 1-m plots, the pooled distribution for 120 cm (figure 22) is again log-normal. Inspection of the pooled scale factor distribution for 120 cm in the plots of the experimental field (figure 9, Part I) reveals the same result. The distributions for the 240 and 300 cm depths in the field, however, appear to be multi-modal.

Initially for $t = 0$, flux given by the drainage equation for unit hydraulic gradient is distributed as K_o, and as time proceeds the skewed flux distribution is cut off by the asymptotic behavior of that equation. Therefore in a similar soil, which is characterized by a single value of β, the distribution becomes narrow and skewness vanishes. This result is observed in figure 26. Thus convergence of the flux distribution to a single value is a distinguishing property for a set of similar soil locations. Apparently, only sets of locations belonging to nonsimilar soil groups, that is, with variable β, can exhibit large flux variations after initial drainage has taken place. Of course, the concept of scaling model functions for soil-water properties is at best an approximation, and common parameters such as β will have statistical variation, which can cause deviations from an ideal behavior. Regarding this possibility, it is of interest to consider the investigations of Warrick, Mullen, and Nielsen (1977*b*) on the prediction of flux, for which a log-

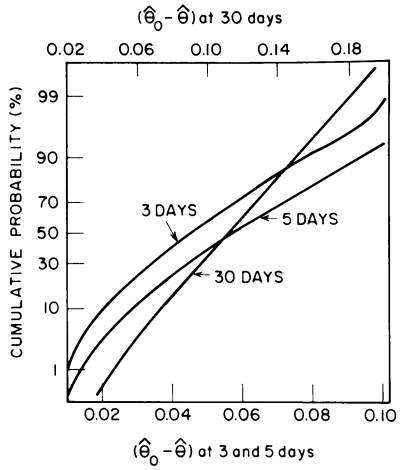

Fig. 25. Cumulative probability of the change in depth-averaged water content at 120 cm for 3, 5, and 30 drainage days, based on distribution of figure 22 for the 1-m plots.

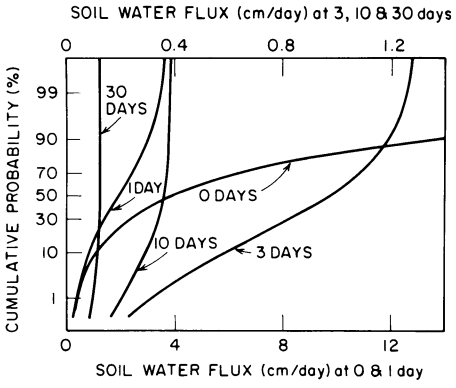

Fig. 26. Cumulative probability of the flux divided by c for 0, 1, 3, 10, and 30 drainage days, based on distribution of figure 22 for the 1-m plots.

normal distribution was found to hold at all times. In contrast, for the 1-m plots the log-normal distribution vanishes within 3 days. Those authors also find that the mean flux is greater than the deterministic value, whereas the opposite holds in figure 24. This difference in results can be attributed to the introduction of the coefficient c; Warrick *et al.* (1977) did not distinguish between water content and the depth-averaged water content. The observed relation of mean flux and measured flux depends on the prevailing pressure head gradient. Evidently, the gradient was positive in the Panoche soil experiment, whereas it was mainly negative in the 1-m plots (except for plot 3). Residual variation of flux after elapse of a sufficient drainage period is due to variation in β, since asymptotically the flux satisfies (133) and becomes distributed essentially as $-\ln \beta$, neglecting variation in c. Inspection of the cumulative probability for β and $\ln \beta$ given by Warrick *et al.* (1977; figure 1) indicates that β is approximately log-normally distributed as well as $\ln \beta$; and moreover, the β distribution is narrow, which suggests that the middle 50 percent of the conductivity sample could be identified with a common mean value of β by neglecting a small error. Indeed, the scaling operation applied to the 1-m plots determines a common representative β—an average value.

Of course, the direct fit of the exponential conductivity model (table 1) indicates that β is actually a random parameter; however, the coefficient of variation is relatively small: c.v. = 0.19 compared with c.v. = 0.9 for the Panoche soil. Now, the coefficients of variation of flux and β are approximately related asymptotically by

$$\sigma_J/\bar{J} = \sigma_\beta/\bar{\beta}, \tag{165}$$

where variation in c has been disregarded. Observations for $\ln J$ (Warrick, Mullen, and Nielsen, 1977*b*; fig. 2) are consistent with equation (165) for 10 days of drainage, and perhaps sooner, but statistics were not provided. Furthermore, cumulative probability for $\ln J$ at 10 days is just a translation in time of the distribution of $\ln \beta$ (Warrick, Mullen and Nielsen, 1977*b*; fig. 1) according to equation (133) with c equal 1.

In conclusion, perhaps a claim that flux is always log-normally distributed is not justified, at least when based on variation of K_0 alone. To understand the origin of variability in β, it is important to note that β depends on the range of measured θ and

the sample size, and is subject to statistical error within the conductivity estimate at each soil location (Fluhler, Ardakani, and Stolzy, 1976). Also, in the case of the 1-m plots, an inadequate determination of the pressure head profile might have caused a reduced estimate for the variance of β. The asymptotic behavior of flux, however, still remains a useful property for testing the validity of scaling when applied to a particular soil.

Summary and Conclusions

Within the limitations of local measurement errors and an approximate evaluation of the pressure head gradient, the scaling of conductivity was adequately achieved with an exponential function of water content. Verification of scaling for the conductivity, however, was not as certain as that for the pressure head. This occurred because estimation of conductivity is confounded by calculational error, whereas pressure head, being a directly measured property, does not include that difficulty. On the other hand, theoretical methods for calculating conductivity from soil pore distributions, such as the Millington-Quirk method, are consistent with the scale relations, since the principles of soil similarity are implicitly included in their derivation. Those methods are particularly useful because conductivity estimates are extended into the dry range of water content. Moreover, by using the scaled soil-water characteristic curves and associated scale factor distribution, the conductivity can be predicted at all locations within a field from measurements taken at only a single location: a spatial extension of conductivity. Application of the Millington-Quirk method with the model soil-water characteristic equation (94) was found to be consistent with an exponential conductivity model. But variation in the steady state infiltration value of water content θ_o, presents a special difficulty to the spatial extension of conductivity, since error in conductivity depends exponentially on the error in θ_o. Therefore an accurate determination of θ_o at each location is required in order to estimate conductivity from its scale relation.

For the 1-m plots drainage experiment, the flux was scaled and scale factors estimated indirectly by scaling the water content profiles. Those results suggested that flux satisfies an altered scale relation, which differs from the original relation and is identical in form to that of the conductivity. All indications are that the method is applicable to any similar drainage experiment involving a region of similar soil. Using the scale factors for flux, the conductivity can be calculated over the entire profile; and the calculation, which is based on an integral of Darcy's law, requires measurements of pressure head at only two extremes of the soil profile. Although the methods that were presented utilized simplified equations, each involving only two parameters, those methods can be generalized for more complex functional relations. For example, a more general drainage equation given by

$$c\,\frac{d\theta}{dt} = J_o \exp\{\sum_{k=1}^{p} \delta_k\,(\theta-\theta_o)^k\} \tag{166}$$

could be used in conjunction with the integral equation

$$\int_{z_1}^{z_2} \frac{J_o}{K_o} \exp\{\sum_{k=1}^{p} (\delta_k-\beta_k)(\theta-\theta_o)^k\}\,dz = z_2 - z_1 + h_2 - h_1 \tag{167}$$

to simultaneously estimate and scale the conductivity, equation (96). Considerably more mathematical detail is involved, but the concepts are not changed.

As an example application of the matching of scale distributions, flux can be estimated for the experimental field at a depth for which conductivity is unknown. An unknown flux J' at depth z' is related to the flux J at depth z and the measured water content profile as follows:

$$J = \frac{\partial}{\partial t} \int_{z}^{z'} \theta \, dz + J', \tag{168}$$

where the depth below the surface is positive and z' greater than z. Of course, this conservation equation must be applied to depths below the sink of crop roots. At each location with scale factor α, J could be obtained as

$$J = \alpha^2 K_m(\theta)(1 + \Delta h/\Delta z) \tag{169}$$

where $K_m(\theta)$ is derived from measurements at depth z in the 1-m plots, and the required pressure head gradient is calculated by usual finite difference techniques using the scaled soil-water characteristics of the experimental field. A best choice for depth z is such that α is least variable over locations. The utility of this method is the reduction in conductivity measurements and the avoidance of a need to estimate evapotranspiration. Indeed, this method allows estimation of the flux using measurements only below the soil surface.

Provided that the fundamental principle of scaling is satisfied, that is, pressure head and conductivity scale factors are essentially equal, the stochastic behavior of water movement through a spatially varying region of similar soil can be characterized completely by a distribution of scale factors. The scaling method could provide a particular advantage if combined with computer simulation models of water movement in combined soil and plant systems. This is so because predictions obtained from deterministic simulation models must be treated as stochastic estimates of actual behavior under field conditions. When scaling is applicable, the number of Monte Carlo computer runs required to represent an average over entire crop seasons can be considerably reduced, since the number of random parameters is reduced. Now, because the eventual fate of fertilizer nitrate in the environment is inseparably linked to variations in water movement, prediction of nitrate movement and best irrigation practice depend on the effects of soil variability. The scaling method, which applies to regions consisting of similar soil, provides a simplified method for dealing with the spatial variability of soil.

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APPENDICES

APPENDIX A

Method to minimize sum of squares and estimate model parameters

A general method for nonlinear least squares fitting which is applicable to scaling is derived below. The function f represents any soil water property or transformation of such a property.

The experimental measurements are x_{ri}, y_{ri} for locations r = 1, ..., R and values i = 1, ..., n_r. Let

$$\hat{y}_r = f(x; a_r, b_1, \ldots, b_m) \qquad (r = 1, \ldots, R) \tag{1}$$

be the assumed functional relation between two properties x and $\hat{y}$. Assume x_{ri} are measured without error and y_{ri} are experimental estimates of $\hat{y}_{ri}$ with error ε_{ri} given by

$$\hat{y}_{ri} = y_{ri} + \varepsilon_{ri} \tag{2}$$

where $E[\varepsilon_{ri}] = 0$, $\text{var}[\varepsilon_{ri}] = \sigma^2_{ri}$, and $\text{Cov}[\varepsilon_{ri}, \varepsilon_{rj}] = 0$.

Assuming a normal distribution of errors, the minimum function (Chi-square) by the method of maximum likelihood is

$$M = \sum_{r=1}^{R} \sum_{i=1}^{n_r} \varepsilon^2_{ri}/\sigma^2_{ri} \tag{3}$$

or

$$M = \sum_{r=1}^{R} \sum_{i=1}^{n_r} [f(x_{ri}; a_r, b) - y_{ri}]^2/\sigma^2_{ri} \tag{4}$$

The parameters $a_1, \ldots, a_R$ and common parameters $b \equiv (b_1, \ldots, b_m)$ are estimated such that (4) is minimum.

In general f is not a linear function of its parameters and therefore the following iteration method is employed to minimize (4). The function (1) is linearized about some initial estimates $a^o_1, \ldots, a^o_R$, b^o of the parameters:

$$f(x; a_r, b) = f(x; a^o_r, b^o) + \frac{\partial f}{\partial a}(x; a^o_r, b^o)(a_r - a^o_r) + \sum_{j=1}^{m} \frac{\partial f}{\partial b_j}(x; a^o_r, b^o)(b_j - b^o_j) \tag{5}$$

$$\text{Then} \quad \hat{y}_{ri} = \hat{y}^o_{ri} + A_{ri}(a_r - a^o_r) + B_{ri}(b - b^o) \tag{6}$$

for $(r = 1, \ldots, R)$, $(i = 1, \ldots, n_r)$ where $\hat{y}^o_{ri} = f(x_{ri}; a^o_r, b^o)$

and $A_{ri} = \frac{\partial f}{\partial a_r}(x_{ri}; a^o_r, b^o)$, $B_{rij} = \frac{\partial f}{\partial b_j}(x_{ri}; a^o_r, b^o)$

for $(j = 1, \ldots, m)$ and b in column vector form. Now let

$$\hat{\eta}_{ri} = A_{ri}\,\alpha_r + B_{ri}\,\beta, \tag{7}$$

where $\eta_{ri} = y_{ri} - \hat{y}^o_{ri}$, $\alpha_r = a_r - a^o_r$, and $\beta = b - b^o$.

Then minimize the following approximate minimum function:

$$M = \sum_{r=1}^{R} \sum_{i=1}^{n_r} g_{ri}(\hat{\eta}_{ri} - \eta_{ri})^2 \tag{8}$$

where $g_{ri} = 1/\sigma^2_{ri}$. The parameters $\alpha_1, \ldots, \alpha_R$, and β are determined by the conditions

$$\frac{\partial M}{\partial \alpha_r} = 0 \quad (r = 1, \ldots, R) \text{ and } \frac{\partial M}{\partial \beta_j} = 0 \quad (j = 1, \ldots, m) \tag{9}$$

Conditions (9) give

$$\sum_{i=1}^{n_r} A_{ri}(A_{ri}\,\alpha_r + B_{ri}\,\beta - \eta_{ri})\,g_{ri} = 0 \quad (r = 1, \ldots, R) \tag{10}$$

and

$$\sum_{r=1}^{R} \sum_{i=1}^{n_r} B_{rij}(A_{ri}\,\alpha_r + B_{ri}\,\beta - \eta_{ri})\,g_{ri} = 0 \quad (j = 1, \ldots, m) \tag{11}$$

Define

$$C_r = \sum_{i=1}^{n_r} A^2_{ri}\,g_{ri} \qquad (r = 1, \ldots, R)$$

$$D_{rj} = \sum_{i=1}^{n_r} A_{ri}\,B_{rij}\,g_{ri} \qquad (j = 1, \ldots, m)$$

$$F_{rjk} = \sum_{i=1}^{n_r} B_{rij}\,B_{rik}\,g_{ri} \qquad (j\ k = 1, \ldots, m)$$

$$G_r = \sum_{i=1}^{n_r} A_{ri}\,\eta_{ri}\,g_{ri} \qquad (r = 1, \ldots, R)$$

$$H_{rj} = \sum_{i=1}^{n_r} B_{rij}\,\eta_{ri}\,g_{ri} \qquad (j = 1, \ldots, m) \tag{12}$$

Then the following system of equations is solved:

$$C_r\,\alpha_r + D^T_r\,\beta = G_r \quad (r = 1, \ldots, R) \tag{13}$$

$$\sum_{r=1}^{R} D_r \alpha_r + \sum_{r=1}^{R} F_r \beta = \sum_{r=1}^{R} H_r \quad (mxm) \tag{14}$$

Substitution of α_r from equation (13) into (14) yields

$$\sum_{r=1}^{R} \left[F_r - \frac{D_r D_r^T}{C_r} \right] \beta = \sum_{r=1}^{R} \left[H_r - \frac{G_r D_r}{C_r} \right] \tag{15}$$

Equation (15) is an mxm system of linear equations for β. With β determined, α_r are found by back solving equations (13).

Special methods are required to determine an initial estimate of the parameters $a_1^o, \ldots, a_R^o, b^o$. These depend on the particular function f. The corrected parameters are given by

$$a_r = a_r^o + \alpha_r \qquad (r = 1, \ldots, R)$$
$$b = b^o + \beta. \tag{16}$$

Iterations using the corrected parameters (16) as new initial parameters are continued according to the scheme until a desired accuracy is obtained. The convergence of this iteration method depends on the function f and initial parameter estimates.

Special method to obtain initial parameter estimates

The soil water characteristic model function is

$$h(s) = a(e^{b(s-1)} - 1). \tag{17}$$

Using second order Taylor series expansions, the model (17) has the following approximate equations:

$$h(s) = a[b(s-1) + b^2(s-1)^2/2] \tag{18}$$

and

$$s(h) = \frac{1}{b} [h/a - h^2/2a^2] + 1. \tag{19}$$

Estimates of a_r and b at each location can be obtained by standard polynomial regression methods using either (18) or (19). An average common b can be taken as initial common parameter. Also, initial estimates of a_r and b can be obtained by using the above iteration method with the approximate model function (19) and "any" initial estimates for the parameters. In fact, equation (19) was found to provide as an acceptable model for soil water characteristics, even though as an approximate series expansion it does not correspond to the model (17) with equal parameters. The convergence, of course, depends on the range of h.

Best scaling results were obtained by taking saturation s as the dependent variable y and pressure head h as the independent variable x. The convergence of the iteration method is strong for the model (17). Equal a_r are taken as initial estimates and the values of a_r and b need only be of proper order of magnitude, provided they are within the domain of the model function (17). Variances were assumed equal: $\sigma_{ri}^2 = \sigma_s^2$; and the minimum sum of squares of deviation in saturation at each location were used to provide an estimate of variance σ_s^2 at each location, in a way similar to standard regression techniques Computer programs provided in the Appendices are in Fortran IV language adapted for a Burroughs B6700/B7700 computer system. Plot subroutines are those implemented by the University of California Computer Center. All programs use data input from disk files.

GENERAL SCALING PROGRAM

```
100 $RESET FREE
200 $SET SEPARATE
300 $SET LIBRARY
400         SUBROUTINE SCALE(N,M,NR,X,Y,A,B,NCOEF,G,ITMAX,BEROR)
500         DIMENSION X(M),Y(M),G(M),NR(N),A(N),B(NCOEF)
600         DIMENSION AO(100),BO(5),D(100,5),E(100)
700         DIMENSION SE(100),SEO(100),NCUM(101)
800         DIMENSION H(5),Z(5),R(5,5),AMAT(5,6),BVEC(5)
1000 C       ***************************************************
1100 C    CHI SQ. FIT OF Y=F(X;A(K),B) FOR K=1 TO N GROUPS OF
1200 C    DATA (X,Y).  REQUIRES FUNCTION F(X;A,B) AND DERIVATIVES
1300 C    GIVEN AS SUBPROGRAM DERIV.  B(NCOEF) ARE COMMON COEFFS.
1400 C    INITIALLY  G IS THE STD. ERROR OF Y GIVEN FOR EACH
1500 C    GROUP.  NUMBER OF ITERATIONS IS ITMAX.
1600 C    NR IS THE NUMBER OF DATA POINTS IN EACH GROUP
1700 C      ***************************************************
1800          ITMAX=ITMAX+1; BEROR0=BEROR
1900          DO 100 K=1,N; NCUM(K+1)=NCUM(K)+NR(K)
2000          DO 100 I=NCUM(K)+1,NCUM(K+1); STDE=G(I)
2100          G(I)=1.; IF(STDE.EQ.0) GO TO 100
2200          G(I)=1./G(I)**2
2300    100   CONTINUE
2350 C     WEIGHT  EQUALS 1 IF STD ERRORS  ARE ZERO
2400       1  IT=IT+1; CHI=0; SSQ=0
2500          DO 2 I=1,NCOEF ;BVEC(I)=0; DO 2 J=1,NCOEF
2600       2  AMAT(I,J)=0
2700          DO 10 K=1,N; C=0; E(K)=0; SE(K)=0
2800          DO 3 I=1,NCOEF; D(K,I)=0; H(I)=0; DO 3 J=1,NCOEF
2900       3  R(I,J)=0
3000          DO 5 I=NCUM(K)+1, NCUM(K+1)
3100          CALL DERIV(X(I),A(K),B,F,W,Z)
3200          F=Y(I)-F; FF=F*F; SE(K)=SE(K)+FF
3300          CHI=CHI+G(I)*FF; GW=G(I)*W
3400          C=C+GW*W; E(K)=E(K)+GW*F
3500          DO 4 II=1,NCOEF; H(II)=H(II)+Z(II)*F*G(I)
3600          D(K,II)=D(K,II)+GW*Z(II)
3700          DO 4 JJ=1,NCOEF
3800       4  R(II,JJ)=R(II,JJ)+Z(II)*Z(JJ)*G(I)
3900       5  CONTINUE
4000          E(K)=E(K)/C; SSQ=SSQ+SE(K)
4100          DO 6 I=1,NCOEF; DC=D(K,I)/C
4200          BVEC(I)=BVEC(I)+(H(I)-D(K,I)*E(K))
4300          DO 6 J=1,NCOEF
4400       6  AMAT(I,J)=AMAT(I,J)+(R(I,J)-DC*D(K,J))
4500          DO 7 I=1,NCOEF
```

```
4600      7  D(K,I)=D(K,I)/C
4700     10  CONTINUE
4800 C   **  CORRECT COEFFS. AND STORE ITERATION RESULTS.
4900         IF(IT.EQ.1)GO TO 15; IF(CHI.GT.CHIO) GO TO 18
5000     15  ITO=IT-1; SSQO=SSQ; CHIO=CHI
5100         DO 16 I=1,NCOEF
5200     16  BO(I)=B(I)
5300         DO 17 K=1,N; SEO(K)=SE(K)
5400     17  AO(K)=A(K)
5500     18  IF(IT.EQ.ITMAX)GO TO 25
5600         IF(BEROR.LT.BERORO)GO TO 25
5700         CALL SOLVE(AMAT,BVEC,Z,NCOEF)
5800         BEROR=0; DO 19 J=1,NCOEF; BEROR=BEROR+ABS(Z(J)/B(J))
5900     19  B(J)=B(J)+Z(J); BEROR=100*BEROR/NCOEF
6000         DO 21 K=1,N; DA=0; DO 20 J=1,NCOEF
6100     20  DA=DA+D(K,J)*Z(J) ; DA=E(K)-DA
6200     21  A(K)=A(K)+DA
6300         IF(IT.EQ.1)PRINT/,'INITIAL CHI SQ.=',CHI,' TOTAL SSQ=',SSQ
6400         GO TO 1
6500 C   **  PRINT RESULTS OF ITERATIONS.
6600     25  SSQ=SSQO; CHI=CHIO
6700         DO 26 J=1,NCOEF
6800     26  B(J)=BO(J)
6900         PRINT/,ITO,'ITERATIONS: FINAL CHI SQ.=',CHI,' TOTAL SSQ=',SSQ
7000         PRINT/,' '
7100         PRINT/,'NUMBER OF GROUPS=',N
7200         PRINT/,'NUMBER OF DATA POINTS=',NCUM(N+1)
7300         PRINT/,' '
7400         PRINT/,'A COEFFICIENTS AND STD. ERROR FOR EACH GROUP'
7500         NCOEF=NCOEF+1; NT=NCUM(N+1)-N*NCOEF
7600         DO 30 K=1,N; A(K)=AO(K)
7700         SE(K)=SQRT(SEO(K)/(NR(K)-NCOEF))
7800     30  PRINT/,K,A(K),SE(K),'D.F.=',NR(K)-NCOEF
7900         NCOEF=NCOEF-1; PRINT/,' '
8000         PRINT/,'COMMON COEFFS. B =',B
8100         PRINT/,'POOLED STD. ERROR=',SQRT(SSQ/NT),'D.F.=',NT
8200         PRINT/,'ERROR TOLERANCE IN COEFFS. B =',BEROR
8300         RETURN
8400         END
8500        SUBROUTINE SOLVE(A,Y,X,N)
8600 C
8700 C
8800 C      THIS SUBROUTINE SOLVES THE MATRIX EQUATION A*X=Y FOR X WHERE
8900 C      A IS A TWO DIMENSIONAL MATRIX OF ORDER N AND X AND Y ARE
9000 C      ONE DIMENSIONAL VECTORS OF ORDER N
9100 C
9200 C      THE ORIGINAL MATRIX A IS DESTROYED DURING THE SUBROUTINE
9300 C
9400         DIMENSION A(5,6),Y(N),X(N),LOC(5),CK(5)
9500        NP=N+1
9600        DO 1 I=1,N
9700        A(I,NP)=Y(I)
9800    1   CK(I)=0.0
9900        DO 100 I=1,N
10000        IP=I+1
10100 C      FIND MAX ELEMENT IN I-TH COL
10200        AMAX=0.0
10300        DO 2 K=1,N
10400        IF(AMAX-ABS(A(K,I)))3,2,2
10500 C      IS NEW MAX IN ROW PREVIOUSLY USED A PIVOT
10600    3   IF(CK(K)) 4,4,2
10700    4   LOC(I)=K
10800        AMAX=ABS(A(K,I))
10900    2   CONTINUE
11000 C      MAX ELEMENT IN I-TH COL IS A(L,I)
11100    5   L=LOC(I)
11200        CK(L)=1.
11300 C      PERFORM ELIMINATION, L IS PIVOT ROW, A(L,I), IS PIVOT ELEMENT
11400        DO 50 J=1,N
11500        IF(L-J) 6,50,6
11600    6   F=-A(J,I)/A(L,I)
11700        DO 40 K=IP,NP
11800    40  A(J,K)=A(J,K)+F*A(L,K)
11900    50  CONTINUE
12000    100 CONTINUE
12100        DO 200 I=1,N
12200        L=LOC(I)
12300    200 X(I)=A(L,NP)/A(L,I)
12400        RETURN
12500        END
*
```

APPENDIX B. Computer program for scaling pressure head.

```
$RESET FREE
$SET AUTOBIND
$BIND = FROM OBJECT/SUB/STAT, FORTRANLIBRARY
$SET SUPRS
FILE  7(KIND=DISK, FILETYPE=7,TITLE="AVGHTHETA")
FILE  6(KIND=REMOTE, MAXRECSIZE=22)
FILE  4=ONEMCOEFF,UNIT=DISK,SAVE=30
      DIMENSION S(176),H(176),ERR(22),NODE(8),A(8)
      DIMENSION ALPHA(8), TH(8),SE(8)
      DIMENSION XTITLE(6), YTITLE(6)
      COMMON NCUM(100)
      DATA XTITLE(1)/"-HEAD"/,YTITLE(1)/"SAT."/
C **  PROVIDE THE INITIAL PARAMETERS HERE; A(NPLOT) AND Z.
      DATA A/8*-200./
      DATA Z/-1./
C **  A IS GIVEN AS THE B OF PHYSICAL MODEL
      DATA NPLOT,IASK/8,0/
      DATA ITMAX, BEROR/4,0/
      IF(IASK.EQ.1)READ(4,/) NPLOT,D,Z,(A(I),I=1,NPLOT)
C **  PHYSICAL MODEL FUNCTION DEFINED HERE
      SFN(X,B,Z)=1.+ALOG(1.+X/B)/Z
      HFN(X,B,Z)=B*(EXP(Z*(X-1.))-1.)
      SMALL=1
      PRINT/,'LIST OF DATA SCALED'
      DO 2 IPLOT=1,NPLOT
      READ(7,/) NCODE,NDEPTH,THSAT,NODE(IPLOT)
      NCUM(IPLOT+1)=NCUM(IPLOT)+NODE(IPLOT)
      K1=NCUM(IPLOT)+1; K2=NCUM(IPLOT+1)
      READ(7,/) (S(I),I=K1,K2)
      READ(7,/) (ERR(I),I=1,NODE(IPLOT))
      READ(7,/) (H(I),I=K1,K2)
      READ(7,/) (ERR(I),I=1,NODE(IPLOT))
      IF(TH(IPLOT).NE.0) THSAT=TH(IPLOT)
      SLOW=1;SHIGH=0;HLOW=1000.;HHIGH=0
      DO 1 I=K1,K2; S(I)=S(I)/THSAT
      HH=H(I); SS=S(I)
      IF(SS.LT.SLOW)SLOW=SS;IF(SS.GT.SHIGH)SHIGH=SS
      IF(HH.LT.HLOW)HLOW=HH;IF(HH.GT.HHIGH)HHIGH=HH
1     CONTINUE
      IF(SLOW.LT.SMALL)SMALL=SLOW
      A(IPLOT)=-1./(Z*A(IPLOT))
      IF(IASK.GT.0) GO TO 2
      NR=NODE(IPLOT)
      PRINT/,IPLOT,'LOCATION',NCODE,'DEPTH',NDEPTH,'SAT. THETA',THSAT
      PRINT/,'SATURATION LIMITS   ',SLOW,'TO',SHIGH
      PRINT/,'PRESSURE HEAD LIMITS ',HLOW,'TO',HHIGH,'NUMBER=',NR
      PRINT/,'  '
2     CONTINUE
      NTOTAL=NCUM(NPLOT+1)
      PRINT/,'TOTAL DATA POINTS SCALED =',NTOTAL
      CLOSE 7
      PRINT/,'*****************************'
      PRINT/,'PHYSICAL MODEL SCALED'
      PRINT/,'FOR SOIL WATER CHARACTERISTICS'
      PRINT/,' PHYSICAL MODEL IS'
      PRINT/,'    S-1=1/Z LOG(1+H/B)'
      PRINT/,'*****************************'
      PRINT/, '*  *  *  *  *  *  *'
      PRINT/,'STANDARD ERROR IN SATURATION'
      CALL SCALE(NPLOT,NTOTAL,NODE,H,S,A,Z,SE,ITMAX,BEROR)
      PRINT/,' *  *  *  *  *  *  *'
      PRINT/,'COEFFS. FOR EQ. S-1=1/Z LOG(1+H/B)'
      DO 5 IPLOT=1,NPLOT
      A(IPLOT)=-1./(Z*A(IPLOT)); B=A(IPLOT)
      BM=BM+1./B; BAVG=BAVG+B
5     CONTINUE
      PRINT/,'Z=',Z
      PRINT/,'B=',(A(I),I=1,NPLOT)
C ** COMPUTE SCALE FACTORS AND SCALE MEAN FUNCTION
      R=NPLOT
      BM=R/BM; BAVG=BAVG/R; FAC=BAVG/BM
      PRINT/,'SCALE MEAN FUNCTION COEFFICIENTS'
      PRINT/,'BM=',BM,'Z=',Z,'AVG. B=',BAVG
      WRITE(4,/) NPLOT,BM,Z,(A(I),I=1,NPLOT)
      PRINT/,' ';PRINT/,'? ? ? ? ? ? ? ?'
      PRINT/,'****** GIVE AN IASK VALUE FOR PLOTS: 0 , 1 , 2'
      PRINT/,'        INPUT  -1  TO EXIT'
```

```
7600          READ/, IASK
7700          IF(IASK.LT.0)GO TO 50
7800          IF(IASK.LT.1) GO TO 12
7900          INT=10.*SMALL; SMIN=INT/10.
8000          INT=10-INT; ISPACE=INT*10
8100          CALL HASH;CALL ISETIT(100,ISPACE,10,INT,0,250.,SMIN,1.)
8200    12    PRINT/,'SCALE FACTORS AND  STD. ERRORS'
8300          PRINT/,'STANDARD ERROR IN PRESSURE HEAD'
8400          DO 15 K=1,NPLOT; NR=NODE(K)
8500          IF(IASK.EQ.3) BM=BAVG
8600          ALPHA(K)=BM/A(K)
8700          K1=NCUM(K)+1; K2=NCUM(K+1)
8800          SSE=0; DO 14 I=K1,K2
8900          HD=-HFN(S(I),A(K),Z)
9000    14    SSE=SSE+(HD-H(I))**2
9100          STDE=SQRT(SSE/(NR-2)); SE(K)=STDE
9200          SSQ=SSQ+SSE*ALPHA(K)**2
9300          IF(IASK.LT.1) GO TO 15
9400          DO 15 I=K1,K2; HA=ALPHA(K)*H(I)
9500          CALL DATAIN('*',HA,S(I))
9600    15    CONTINUE
9700          PRINT/,' '
9800          PRINT/,'SCALES ALPHA=',(ALPHA(I),I=1,NPLOT)
9900          PRINT/,' '
10000          PRINT/,'STD. ERRORS=',(SE(I),I=1,NPLOT)
10100          WRITE(4,/)  (SE(I),I=1,NPLOT)
10200          LOCK 4
10300          PRINT/,'  '
10400          PRINT/,'SUM SQ. OF DEV. FOR SCALE MEAN FUNCTION',SSQ
10500          PRINT/, 'STD. ERROR=',SQRT(SSQ/(NTOTAL-2)),'D.F.=',NTOTAL-2
10600          IF(IASK.LT.1)GO TO 50
10700          DO 20 I=1,9; HD=25.*I
10800          SAT=SFN(-HD,BM,Z)
10900    20    CALL DATAIN('C',HD,SAT)
11000          PRINT/,' '; PRINT/,'  '
11050          PRINT/,'HOLD FOR PLOT'; READ/,HOLD
11100          PRINT/,' PLOT OF SCALED DATA'; PRINT/,'
11200          CALL PRNTIT(2,2,XTITLE,5,YTITLE,4,6)
11300          IF(IASK.LT.2)GO TO 50
11400          CALL HASH;CALL ISETIT(100,ISPACE,10,INT,0,250,SMIN,1.)
11500          DO 30 IPLOT=1,NPLOT; DO 30 I=NCUM(IPLOT)+1,NCUM(IPLOT+1)
11600    30    CALL DATAIN('*',H(I),S(I))
11700          DO 40 IPLOT=1,NPLOT
11800          DO 40 I=1,9; HD=25.*I
11900          SAT=SFN(-HD,A(IPLOT),Z)
12000    40    CALL DATAIN('C',HD,SAT)
12100          PRINT/,'***********************************************'
12200          PRINT/,'PLOT OF DATA AT EACH LOCATION';PRINT/,' '
12300          CALL PRNTIT(2,2,XTITLE,5,YTITLE,4,6)
12400    50     STOP
12500          END
12600 C ** DERIVATIVES OF  PHYSICAL MODEL
12700 C **  SPECIAL TRANSFORMATION A=1/ZB.
12800          SUBROUTINE DERIV(H,A,Z,S,DSDA,DSDZ)
12900          AZH1=A*Z*H+1.; S=ALOG(AZH1)/Z
13000          DSDA=H/AZH1
13100          DSDZ=(-S+A*DSDA)/Z; S=S+1.
13200          RETURN
13300          END
*


L
100 $RESET FREE
200 $SET SEPARATE
300 $SET LIBRARY
400 C ** LEAST SQ. FIT OF Y=F(X;A(K),B) FOR K=1 TO N GROUPS OF
500 C ** DATA (X,Y). REQUIRES FUNCTION F(X;A,B) AND DERIVATIVES
600 C ** GIVEN AS SUBROUTINE DERIV.  B IS COMMON COEFF.
700         SUBROUTINE SCALE(N,M,NR,U,Y,A,B,SE,ITMAX,BEROR)
800         DIMENSION U(M),Y(M),NR(N),A(N),SE(N)
900         DIMENSION AO(100),SEO(100),C1(100),C2(100)
1000          COMMON NCUM(100)
1100          ITMAX=ITMAX+1; BEROR=BERORO
1200     1    IT=IT+1;S1=0;S2=0;SSQ=0
1300          DO 3 K=1,N
1400          SUMXF=0;SUMZF=0;SUMXX=0;SUMZZ=0;SUMXZ=0
1500          SE(K)=0; DO 2 I=NCUM(K)+1,NCUM(K+1)
1600          CALL DERIV(U(I),A(K),B,F,X,Z)
1700          F=Y(I)-F; SE(K)=SE(K)+F**2
```

```
1800          SUMXF=SUMXF+X*F; SUMZF=SUMZF+Z*F
1900          SUMXX=SUMXX+X*X; SUMZZ=SUMZZ+Z*Z
2000     2    SUMXZ=SUMXZ+X*Z
2100          C1(K)=SUMXZ/SUMXX; C2(K)=SUMXF/SUMXX
2200          S1=S1+SUMZZ-SUMXZ*C1(K)
2300          S2=S2+SUMZF-SUMXZ*C2(K)
2400          SSQ=SSQ+SE(K)
2500     3    CONTINUE
2600          IF(IT.EQ.1) GO TO 4
2700          IF(SSQ.GT.SSQO) GO TO 6
2800     4    ITO=IT-1; SSQO=SSQ; BO=B
2900          DO 5 K=1,N; SEO(K)=SE(K)
3000     5    AO(K)=A(K)
3100     6    IF(IT.EQ.ITMAX) GO TO 8
3200          IF(BEROR.LT.BERORO) GO TO 8
3300          IF(IT.EQ.1) PRINT/,'INITIAL SSQ=',SSQ
3400          DB=S2/S1;IF(B.NE.0)BEROR=ABS(100*DB/B);B=B+DB
3500          DO 7 K=1,N; DA=C2(K)-DB*C1(K)
3600     7    A(K)=A(K)+DA
3700          GO TO 1
3800 C ************************************************
3900 C ** PRINT RESULTS OF ITERATIONS
4000     8    SSQ=SSQO; B=BO
4100          PRINT/,'FINAL SSQ=',SSQ,' FOR ITERATIONS=',ITO
4200          PRINT/,' A COEFFICIENTS AND STD. ERROR FOR EACH GROUP'
4300          DO 9 K=1,N; NT=NT+NR(K)-2
4400          SE(K)=SQRT(SEO(K)/(NR(K)-2)); A(K)=AO(K)
4500     9    PRINT/,K,A(K),SE(K),'D.F.=',NR(K)-2
4600          PRINT/,' COMMON COEFF. B=',B
4700          PRINT/,' POOLED STD. ERROR=',SQRT(SSQ/NT)
4800          PRINT/,' ERROR TOLERANCE IN B =',BEROR
4900          RETURN
5000          END
$
```

APPENDIX C.
Computer program for scaling hydraulic conductivity.

```
100 $RESET FREE
200 $SET AUTOBIND
300 $BIND = FROM OBJECT/SUB/STAT, FORTRANLIBRARY
400 $SET SUPRS
500 FILE  7(KIND=DISK, FILETYPE=7,TITLE="CONDK")
600 FILE  6(KIND=REMOTE, MAXRECSIZE=22)
700 FILE  4=ONEMKCOEFF,UNIT=DISK,SAVE=30
800          DIMENSION S(440),C(440),Y(440),NODE(20),A(20)
900          DIMENSION ALPHA(20),TH(20),SE(20)
1000          DIMENSION XTITLE(6), YTITLE(6)
1100          COMMON NCUM(100)
1200          DATA XTITLE(1)/"LOG K"/,YTITLE(1)/"SAT."/
1300 C **  SPECIAL CASE: LINEAR REGRESSION MODEL
1400 C **  SOLUTION IS EXACT - REQUIRES ONLY ONE ITERATION.
1500 C **  A IS GIVEN AS KO OF CONDUCTIVITY MODEL.
1600          DATA NPLOT, IASK/20,0/
1700          DATA ITMAX, BEROR/1,0/
1900 C **  PHYSICAL MODEL FUNCTION DEFINED HERE
2000          SFN(X,A,B)=1.+ALOG(X/A)/B
2100          CFN(X,A,B)=A*EXP(B*(X-1))
2200          SMALL=1
2300          PRINT/,'LIST OF DATA SCALED'
2400          DO 2 IPLOT=1,NPLOT
2500          READ(7,/) NCODE,NDEPTH,THSAT,NODE(IPLOT)
2600          NCUM(IPLOT+1)=NCUM(IPLOT)+NODE(IPLOT)
2700          K1=NCUM(IPLOT)+1; K2=NCUM(IPLOT+1)
2800          READ(7,/) (S(I),I=K1,K2)
2900          READ(7,/) (C(I),I=K1,K2)
3000          IF(TH(IPLOT).NE.0) THSAT=TH(IPLOT)
3100          SLOW=1;SHIGH=0;CLOW=100.;CHIGH=0
3200          DO 1 I=K1,K2; S(I)=S(I)/THSAT
3300          CC=C(I);SS=S(I)
3400          IF(SS.LT.SLOW)SLOW=SS;IF(SS.GT.SHIGH)SHIGH=SS
3500          IF(CC.LT.CLOW)CLOW=CC;IF(CC.GT.CHIGH)CHIGH=CC
3550          Y(I)=ALOG(C(I))
3600     1    CONTINUE
3700          IF(SLOW.LT.SMALL)SMALL=SLOW
3800          IF(IASK.GT.0) GO TO 2
```

```
3900         NR=NODE(IPLOT)
4000         PRINT/,'LOCATION',IPLOT,' DEPTH',NDEPTH,'SAT. THETA',THSAT
4100         PRINT/,'SATURATION LIMITS    ',SLOW,'TO',SHIGH
4200         PRINT/,' CONDUCTIVITY LIMITS',CLOW,'TO',CHIGH,'NUMBER=',NR
4300         PRINT/,'
4400  2      CONTINUE
4500         NTOTAL=NCUM(NPLOT+1)
4600         PRINT/,'TOTAL DATA POINTS SCALED =',NTOTAL
4700         CLOSE 7
4800         PRINT/,'*******************************'
4900         PRINT/,' MODEL CONDUCTIVITY SCALED'
5000         PRINT/,' PHYSICAL MODEL IS'
5100         PRINT/,'     K=KO EXP(B(S-1))'
5200         PRINT/,'*******************************'
5215 C   ** REGRESSION VARIABLE IS Y=LOG K.
5225 C **   SPECIAL TRANSFORMATION FOR LOG VARIABLES.
5300         PRINT/, '*  *  *  *  *  *  *'
5400         PRINT/,' STANDARD ERROR IN LOG K'
5500         CALL SCALE(NPLOT,NTOTAL,NODE,S,Y,A,B,SE,ITMAX,BEROR)
5600         PRINT/,' *  *  *  *  *  *'
5625         DO 200 K=1,NPLOT
5650   200   A(K)=EXP(A(K))
5700         PRINT/,' COEFFS. FOR EQ. LOG K=B(S-1)+LOG A'
5800         PRINT/,'B=',B
5900         PRINT/,'A=',(A(I),I=1,NPLOT)
6000 C ** COMPUTE SCALE FACTORS AND SCALE MEAN FUNCTION
6100         DO 5 IPLOT=1,NPLOT
6200         AM=AM+SQRT(A(IPLOT)); AVGA=AVGA+A(IPLOT)
6300   5     CONTINUE
6500         R=NPLOT;AM=(AM/R)**2;AVGA=AVGA/R
6600         PRINT/,'SCALE MEAN FUNCTION COEFFICIENTS'
6700         PRINT/,' AM=',AM,'B=',B,'AVG. A=',AVGA
6800         WRITE(4,/) NPLOT,AM,B,(A(I),I=1,NPLOT)
6900         PRINT/,' ';PRINT/,'? ? ? ? ? ? ? ?'
7000         PRINT/,'****** GIVE AN IASK VALUE FOR PLOTS: 0 , 1 , 2'
7100         PRINT/,'        INPUT  -1  TO EXIT'
7200         READ/, IASK
7300         IF(IASK.LT.0)GO TO 50
7400         IF(IASK.LT.1) GO TO 12
7500         INT=10.*SMALL; SMIN=INT/10.
7600         INT=10-INT; ISPACE=INT*10
7700         CALL HASH;CALL ISETIT(80,ISPACE,8,INT,-2,2,SMIN,1.)
7800   12    PRINT/,'SCALE FACTORS AND  STD. ERRORS'
7900         PRINT/,'STANDARD ERROR IN SATURATION'
8000         DO 15 K=1,NPLOT; NR=NODE(K)
8100         IF(IASK.EQ.3) AM=AVGA
8200         ALPHA(K)=SQRT(A(K)/AM)
8300         K1=NCUM(K)+1; K2=NCUM(K+1)
8400         SSE=0; DO 14 I=K1,K2
8500         SAT=SFN(C(I),A(K),B)
8600   14    SSE=SSE+(SAT-S(I))**2
8700         STDE=SQRT(SSE/(NR-2)); SE(K)=STDE
8800         SSQ=SSQ+SSE
8900         IF(IASK.LT.1) GO TO 15
9000         DO 15 I=K1,K2; COND=C(I)/(ALPHA(K)**2)
9100         COND=ALOG10(COND)
9200         CALL DATAIN('*',COND,S(I))
9300   15    CONTINUE
9400         PRINT/,' '
9500         PRINT/,'SCALES ALPHA=',(ALPHA(I),I=1,NPLOT)
9600         PRINT/,' '
9700         PRINT/,'STD. ERRORS=',(SE(I),I=1,NPLOT)
9800         WRITE(4,/)  (SE(I),I=1,NPLOT)
9900         LOCK 4
10000         PRINT/,'  '
10100         PRINT/,'POOLED STANDARD ERROR IN SATURATION'
10200         PRINT/, 'STD. ERROR=',SQRT(SSQ/(NTOTAL-2)),'D.F.=',NTOTAL-2
10300         IF(IASK.LT.1)GO TO 50
10400         DO 20 I=1,11; SAT=1.-0.05*(I-1)
10500         COND=CFN(SAT,AM,B); COND=ALOG10(COND)
10600   20    CALL DATAIN('C',COND,SAT)
10700         PRINT/,' '; PRINT/,'  '
10800         PRINT/,'HOLD FOR PLOT'; READ/,HOLD
10900         PRINT/,' PLOT OF SCALED DATA'; PRINT/,'  '
11000         CALL PRNTIT(2,2,XTITLE,5,YTITLE,4,6)
11100         IF(IASK.LT.2)GO TO 50
11200         CALL HASH;CALL ISETIT(80,ISPACE,8,INT,-2,2,SMIN,1.)
11300         DO 30 IPLOT=1,NPLOT; DO 30 I=NCUM(IPLOT)+1,NCUM(IPLOT+1)
11400   30    CALL DATAIN('*',ALOG10(C(I)),S(I))
11500         DO 40 IPLOT=1,NPLOT
```

```
11600         DO 40 I=1,11;SAT=1.-0.05*(I-1)
11700         COND=CFN(SAT,A(IPLOT),B); COND=ALOG10(COND)
11800   40    CALL DATAIN('C',COND,SAT)
11900         PRINT/,'************************************************'
12000         PRINT/,'PLOT OF DATA AT EACH LOCATION';PRINT/,' '
12100         CALL PRNTIT(2,2,XTITLE,5,YTITLE,4,6)
12200   50     STOP
12300         END
12400 C ** DERIVATIVES OF  PHYSICAL MODEL
12450 C  **  TRANSFORMATION  Y=LOG K.
12500         SUBROUTINE DERIV(S,A,B,Y,DYDA,DYDB)
12600         DYDB=S-1.
12700         DYDA=1.; Y=B*DYDB+A
12800         RETURN
12900         END
```

APPENDIX D.
Computer program for scaling drainage.

```
100 $RESET FREE
200 $SET AUTOBIND
300 $BIND = FROM OBJECT/SUB/SCALE, FORTRANLIBRARY
400 $SET SUPRS
500 FILE  1=ONEMCONCOEF,UNIT=DISK,SAVE=30
600 FILE  2(KIND=DISK,FILETYPE=7,TITLE="THETAREGCOEFF")
700 FILE  3(KIND=DISK,FILETYPE=7,TITLE="TIMES")
800 FILE  4(KIND=DISK,FILETYPE=7,TITLE="AVGWTCONT")
900 FILE  6(KIND=REMOTE,MAXRECSIZE=22)
1000 FILE  7(KIND=DISK,FILETYPE=7,TITLE="DEPTHAVGTH")
1100        DIMENSION THETA(22),U(22),T(22),DEPTH(8),Z(32)
1200        DIMENSION SE(32),RCOEF(4,8),C(32),THO(32),A(32),ALPHA(32)
1300        DIMENSION X(32,19), Y(32,19)
1400        DIMENSION XTITLE(6),YTITLE(6)
1500 C ** PROVIDE INITIAL PARAMETERS HERE - A IS KO OF MODEL
1600        DATA A,B/32*4.,40/
1650 C ** OPTION TO READ INITIAL VALUES OF PARAMETERS.
1675        READIN="YES"
1700        DATA XTITLE/"TIME  ","(DAYS)",4*" "/,YTITLE/"THETA",5*" "/
1800        DATA DEPTH/15,30,45,60,75,90,105,120/
1825 C ** OPTION TO DO SCALE REGRESSION WITH A FIXED COMMON B.
1850        FIXB="NO"
1900 C ** INDICATE NUMBER OF ITERATIONS AS ITMAX.
2000        ITMAX=5; N=20
2100        NPLOT=4;NDEPTH=8;NTIME=19;NLESS=0
2200 C ** DRAINAGE EQUATION DEFINED HERE
2300        THF(T,THO,A,B)=THO-ALOG(1.+A*B*T)/B
2400        FLXF(T,A,B)=A/(1.+A*B*T)
2500 C ** MODEL CONDUCTIVITY FUNCTION DEFINED HERE
2600        CONF(THETA,THO,CONO,B)=CONO*EXP(B*(THETA-THO))
2700 C -------------------------------------------------------------
2800 C ** INDIRECT SCALING OF CONDUCTIVITY FROM WATER CONTENT PROFILE.
2900 C ** ASSUMES REGRESSION OF DEPTH AVG. THETA ON THETA WITH REGRESSION
3000 C ** COEFFICIENTS C FOR EACH LOCATION AND UNIT HYDRAULIC GRADIENT.
3100 C -------------------------------------------------------------
3200        PRINT/,'************************************************'
3300        PRINT/,' INDIRECT SCALING OF CONDUCTIVITY'
3400        PRINT/,' CONDUCTIVITY MODEL IS'
3500        PRINT/,'     K = KO EXP(B(THETA - THO))'
3600        PRINT/,' DRAINAGE EQUATION IS'
3700        PRINT/,'     THETA=THO - LOG(1+B KO T/C Z)/B'
3800        PRINT/,' DEPTH AVG(THETA-THO) = C*(THETA-THO)'
3900        PRINT/,'************************************************'
4000        PRINT/,' '
4025        IF(READIN.EQ."YES") READ(3,/) ITMAX,FIXB,B,A
4100        READ(3,/) T; CLOSE 3; TO=T(1)
4200        DO 1 I=1,NTIME
4300   1    T(I)=T(I)-TO
4400        READ(2,/) ((RCOEF(I,J),J=1,NDEPTH),I=1,NPLOT); CLOSE 2
4410        CALL HASH;CALL ISETIT(100,40,10,4,0,50,.25,.45)
4500        DO 3 IPLOT=1,NPLOT; DO 3 IDEP=1,NDEPTH
4600        READ(4,5) THETA; READ(4,5) U
4700   5    FORMAT( 11(1X,F4.3) )
4800        IF(IDEP.LE.NLESS) GO TO 3
4900        NLOC=NLOC+1;C(NLOC)=RCOEF(IPLOT,IDEP)
5000        Z(NLOC)=DEPTH(IDEP); THO(NLOC)=THETA(1)
5100        DO 2 I=1,NTIME; X(NLOC,I)=T(I)
```

```
5150          CALL DATAIN('*',T(I),THETA(I))
5200    2     Y(NLOC,I)=THO(NLOC)-THETA(I)
5300    3     CONTINUE
5400          PRINT/,' NUMBER OF LOCATIONS SCALED=',NLOC
5500          PRINT/,' NUMBER OF SAMPLE TIMES=',NTIME
5600          PRINT/,' NUMBER OF DATA POINTS SCALED=',NLOC*NTIME
5700          PRINT/,' DRAINAGE PERIOD IS',T(NTIME),' DAYS'
5800 C ** TRANSFORM INITIAL PARAMETERS  FOR REDUCED TIME.
5900          DO 4 K=1,NLOC
6000    4     A(K)=A(K)/(Z(K)*C(K))
6100 C ** REGRESSION VARIABLE USED IN SCALE IS Y=THO-THETA
6200 C **       Y=LOG(1 + A B T)/B
6300          PRINT/,' * * * * * * * * * *'
6400          PRINT/,' SCALE RESULTS'
6500          PRINT/,' STANDARD ERROR IN WATER CONTENT'
6550          IF(FIXB.EQ.'YES')PRINT/,' B IS FIXED AT INITIAL VALUE'
6600          CALL SCALE(NLOC,NTIME,X,Y,A,B,SE,ITMAX,FIXB)
6700          PRINT/,' * * * * * * * * * *'
6800 C ** CORRECT REG. COEFFS. C FOR TIME DEPENDENCE.
6900          PRINT/,' '
7000          PRINT/,' STATISTICS FOR REGRESSION DEPTH AVG THETA VS TIME'
7100          PRINT/,'  CORRECTED REG. COEFFS. C AT EACH LOCATION'
7200          PRINT/,' '
7300          PRINT 25
7500    25    FORMAT(4X,'C COEF',2X,'NEW C',2X,'ERROR',2X,'STD E',1X,'R COEF')
7600          DO 40 IPLOT=1,NPLOT; DO 40 IDEP=1,NDEPTH
7700          READ(7,5) THETA; READ(7,5) U
7800          IF(IDEP.LE.NLESS) GO TO 40; M=M+1
7850          IF(A(M).EQ.0) GO TO 40
7900          DTHO=THETA(1); DO 30 I=1,NTIME
8000          THETA(I)=DTHO-THETA(I)
8100    30    U(I)=-THF(T(I),0,A(M),B)
8200          CALL CREGS(U,THETA,NTIME,CC,SC,STDE,R)
8300          PRINT 35, M,C(M),CC,SC,STDE,R
8400    35    FORMAT(1X,I2,3(3X,F4.2),3X,F4.3,3X,F4.2)
8500          C(M)=CC
8600    40    CONTINUE
8700          PRINT/,' '
8800          DO 6 K=1,NLOC; AVGTHO=AVGTHO+THO(K);A(K)=C(K)*Z(K)*A(K)
8900          AM=AM+SQRT(A(K)); AVGA=AVGA+A(K)
9000    6     CONTINUE
9100          AM=(AM/NLOC)**2;AVGA=AVGA/NLOC;AVGTHO=AVGTHO/NLOC
9200          PRINT/,' SCALE MEAN CONDUCTIVITY COEFFICIENTS:'
9300          PRINT/,' B=',B,'SCALE MEAN KO=',AM
9400          PRINT/,' AVG. KO =',AVGA,' AVG. THO =',AVGTHO
9500          PRINT/,' '
9600          PRINT/,' KO FOR EACH LOCATION:'
9700          PRINT/, (A(K),K=1,NLOC)
9800          PRINT/,' '
9900 C ** COMPUTE SCALE FACTORS ALPHA  FOR CONDUCTIVITY.
9950 C ** COMPUTE CORRECTED STEADY STATE KO FOR COMMON THO.
10000         DO 7 K=1,NLOC; ALPHA(K)=SQRT(A(K)/AM)
10010         A(K)=A(K)*EXP(B*(AVGTHO-THO(K)))
10020   7     AMNEW=AMNEW+SQRT(A(K))
10200         PRINT/,' SCALE FACTORS ALPHA:'
10300         PRINT/, (ALPHA(K),K=1,NLOC); PRINT/,' '
10305 C ** CORRECT SCALES FOR AN AVERAGE STEADY STATE THETA OVER LOCATIONS
10310         AMNEW=(AMNEW/NLOC)**2
10315         PRINT/,' CORRECTED CONDUCTIVITY SCALES FOR COMMON THO'
10320         PRINT/,' '
10325         PRINT/,' NEW SCALE MEAN KO=',AMNEW,' AVG THO=',AVGTHO
10330         PRINT/,' '
10335         PRINT/,' NEW KO AND SCALES FOR EACH LOCATION'
10340         DO 70 K=1,NLOC;OHMEGA=SQRT(A(K)/AMNEW)
10350   70    PRINT 80, K,A(K),THO(K),OHMEGA
10360   80    FORMAT(2X,I3,2X,F6.2,2X,F4.3,2X,F7.4)
10400         PRINT/,' HOLD FOR PLOT: IASK=0 OR 1'; READ/, IASK
10500         IF(IASK.EQ.0) GO TO 100
10600 C ** STORE SCALED COEFFS.
10700         WRITE(1,/) NLOC,B,AMNEW,(A(K),K=1,NLOC)
10750         WRITE(1,/) (THO(K),K=1,NLOC)
10800         WRITE(1,/) (SE(K),K=1,NLOC); LOCK 1
10820         PRINT/,'
10830         PRINT/,'GRAPH OF WATER CONTENT VS EXPERIMENTAL TIME'
10840         PRINT/,' '
10850         CALL PRNTIT(2,2,XTITLE,12,YTITLE,6,6)
10900         CALL HASH;CALL ISETIT(100,40,10,4,0,50,.25,.45)
11000         ZM=DEPTH(NDEPTH); DO 8 K=1,NLOC; SZ=Z(K)/ZM
11100         FAC=ALPHA(K)**2/(C(K)*SZ)
11200 C ** MACRO. DEPTH SCALE SZ W.R.T. LAST DEPTH.
```

```
11300 C ** REDUCE TIME AND PLOT THETA VS REDUCED TIME.
11400         DO 8 I=1,NTIME
11500         TIME=FAC*X(K,I);TH=AVGTHO-Y(K,I)
11600         CALL DATAIN('*',TIME,TH)
11700    8    CONTINUE
11800         CONO=AM/ZM; DO 9 I=1,12; TIME=2.*(I-1)
11850         IF(I.GT.8) TIME=10.*(I-7)
11900         TH=THF(TIME,AVGTHO,CONO,B)
12000    9    CALL DATAIN('C',TIME,TH)
12010         PRINT/,' HOLD FOR PLOT'; READ/, BLANK
12100         PRINT/,' '
12200         PRINT/,' GRAPH OF WATER CONTENT (THETA) VS REDUCED TIME'
12300         PRINT/,' '
12400         CALL PRNTIT(2,2,XTITLE,12,YTITLE,6,6)
12500         PRINT/,' '
12600         PRINT/,' REDUCED TIME = (ALPHA**2/C SZ) * T'
12700         PRINT/,' DEPTH SCALE SZ=Z/ZM FOR ZM=',ZM,'CM'
12800         PRINT/,' '
12900         PRINT/,' MEAN CONDUCTIVITY AND FLUX AT REDUCED TIMES'
13000         PRINT/,' '
13100         PRINT 10
13200    10   FORMAT(1X,'THETA',7X,'COND',3X,'TIME DAYS',5X,'FLUX')
13300         TH=0; THSTEP=AVGTHO/N; D=B/ZM
13400         DO 20 I=1,N
13500         TH=TH+THSTEP; TIME=2.*I
13600         COND=CONF(TH,AVGTHO,AM,B);FLUX=FLXF(TIME,AM,D)
13700         PRINT 15, TH,COND,TIME,FLUX
13800    15   FORMAT(2X,F4.3,2X,E9.3,' | ',5X,F4.1,1X,E9.3)
13900    20   CONTINUE
14000    100  STOP; END
14100 C ** DERIVATIVES OF DRAINAGE EQUATION.
14200         SUBROUTINE DERIV(X,A,B,Y,DYDA,DYDB)
14250         IF(A.LT.0)A=0
14300         F=1.+A*B*X; Y=ALOG(F)/B
14400         DYDA=X/F; DYDB=(A*DYDA-Y)/B
14500          RETURN
14600          END
$
100 $RESET FREE
200 $SET SEPARATE
300 $SET LIBRARY
400 C ** LEAST SQ. FIT OF Y=F(X;A(K),B) FOR K=1 TO N GROUPS OF
500 C ** DATA (X,Y). REQUIRES FUNCTION F(X;A,B) AND DERIVATIVES
600 C ** GIVEN AS SUBROUTINE DERIV.  B IS COMMON COEFF.
700         SUBROUTINE SCALE(N,M,S,Y,A,B,SE,ITMAX,FIXB)
800         DIMENSION S(N,M),Y(N,M),A(N),SE(N)
900         DIMENSION A0(50),SE0(50),C1(50),C2(50)
1000         ITMAX=ITMAX+1
1100    1    IT=IT+1;S1=0;S2=0;SSQ=0
1200         DO 3 K=1,N
1300         SUMXF=0;SUMZF=0;SUMXX=0;SUMZZ=0;SUMXZ=0
1400         SE(K)=0; DO 2 I=1,M
1500         CALL DERIV(S(K,I),A(K),B,F,X,Z)
1600         F=Y(K,I)-F; SE(K)=SE(K)+F**2
1700         SUMXF=SUMXF+X*F; SUMZF=SUMZF+Z*F
1800         SUMXX=SUMXX+X*X; SUMZZ=SUMZZ+Z*Z
1900    2    SUMXZ=SUMXZ+X*Z
2000         C1(K)=SUMXZ/SUMXX; C2(K)=SUMXF/SUMXX
2100         S1=S1+SUMZZ-SUMXZ*C1(K)
2200         S2=S2+SUMZF-SUMXZ*C2(K)
2300         SSQ=SSQ+SE(K)
2400    3    CONTINUE
2500         IF(IT.EQ.1) GO TO 4
2600         IF(SSQ.GT.SSQ0) GO TO 6
2700    4    ITO=IT-1; SSQ0=SSQ; B0=B
2800         DO 5 K=1,N; SE0(K)=SE(K)
2900    5    A0(K)=A(K)
3000    6    IF(IT.EQ.ITMAX) GO TO 8
3100         IF(IT.EQ.1) PRINT/,'INITIAL SSQ=',SSQ
3200         DB=S2/S1;IF(B.NE.0)BEROR=100*DB/B
3250         IF(FIXB.EQ."YES") DB=0
3275         B=B+DB
3300         DO 7 K=1,N; DA=C2(K)-DB*C1(K)
3400    7    A(K)=A(K)+DA
3500         GO TO 1
3600 C ***************************************************
3700 C ** PRINT RESULTS OF ITERATIONS
3800    8    SSQ=SSQ0; B=B0
3900         PRINT/,'FINAL SSQ=',SSQ,' FOR ITERATIONS=',ITO
4000         PRINT/,' A COEFFICIENTS AND STD. ERROR FOR EACH GROUP'
```

```
4100          DO 9 K=1,N; NT=NT+M-2
4200          SE(K)=SQRT(SEO(K)/(M-2)); A(K)=AO(K)
4300     9    PRINT/,K,A(K),SE(K),'D.F.=',M-2
4400          PRINT/,' COMMON COEFF. B=',B
4500          PRINT/,' POOLED STD. ERROR=',SQRT(SSQ/NT)
4600          PRINT/,' ERROR TOLERANCE IN B =',BEROR
4700          RETURN
4800          END
5000 C ** SOLVES LINEAR REGRESSION Y=BX.
5100          SUBROUTINE CREGS(X,Y,N,B,SB,SE,R)
5200          DIMENSION X(N), Y(N)
5300          SUMXY=0;SUMXX=0;SUMYY=0
5400          DO 1 I=1,N; SUMXY=SUMXY+X(I)*Y(I)
5500          SUMXX=SUMXX+X(I)**2
5600     1    SUMYY=SUMYY+Y(I)**2
5700          B=SUMXY/SUMXX
5800          SSDEVR=SUMYY-B*SUMXY
5900          SE=SQRT(SSDEVR/(N-1)); SB=SE/SQRT(SUMXX)
6000          R=SUMXY/SQRT(SUMXX*SUMYY)
6100          RETURN
6200          END
*
```

APPENDIX E.

Computer program for Millington-Quirk hydraulic conductivity.

```
L
100 $SET AUTOBIND
200 $BIND = FROM OBJECT/POLYFIT
300 $SET SUPRS
400 $RESET FREE
500 C-----------------------------------------------------------------
600 C  PROGRAM CALCULATES MILLINGTON-QUIRK CONDUCTIVITY AND MATCHING
700 C  FACTOR FOR EXPERIMENTAL DATA GIVEN AS MODEL CONDUCTIVITY
800 C  FUNCTION.  SUCTION HEADS CORRESPONDING TO PORE CLASSES ARE
900 C  DETERMINED FROM SOIL-WATER CHARACTERISTIC MODEL.
1000 C  MODEL SOIL-WATER CHARACTERISTIC IS
1100 C        H=A(EXP(RHO(THETA-THSAT)) -1).
1200 C  MODEL CONDUCTIVITY FUNCTION IS
1300 C         K=KO EXP(BETA(THETA-THO)).
1400 C  PARAMETERS OF MODELS ARE REQUIRED AS INPUT DATA.
1500 C-----------------------------------------------------------------
1600 FILE  1(KIND=DISK,FILETYPE=7,TITLE='MQCOND')
1700          DIMENSION THETA(50),H(50),WK(50),RSQ(50)
1800          DIMENSION U(50),V(50),B(10),BETA(10)
1900 C-----------------------------------------------------------------
2000 C ** INPUT DATA REQUIRED:
2100 C  NUMBER OF PORE CLASSES-N. SATURATED THETA-THSAT.
2200 C  UPPER LIMIT TO THETA FOR CLASS WITH LARGEST PORE RADIUS-THFULL
2300 C  (PORE RADIUS IS ZERO FOR THETA BELOW THLOW)
2400 C-----------------------------------------------------------------
2500          DATA N,THSAT,THFULL,THLOW/20,.42,.42,.1/
2600 C-----------------------------------------------------------------
2700 C  STEADY STATE INFILTRATION THETA-THO.
2800 C  LOWER LIMIT THETA TO K DATA-THKMIN.
2900 C  LOWER THETA LIMIT FOR OUTPUT OF COMPUTED K -THOUT.
3000 C-----------------------------------------------------------------
3100          DATA THO,THKMIN,THOUT/.42,.30,.20/
3200 C ** CONDUCTIVITY AND SUCTION HEAD MODEL PARAMETERS GIVEN HERE.
3300 C ** B IS THE BETA COEFF. OF K MODEL.
3400          DATA CONO,B(1),A,RHO,/12.72,13.0,114.0,-4.93/
3500          RHO=RHO/THSAT; B(1)=B(1)/THO
3600 C-----------------------------------------------------------------
3700 C  SET DEGREE OF POLYNOMIAL FIT TO M-Q CALC. LOG K.
3750 C  NB NUMBER OF K MODEL COEFFS. B REQUIRED ON INPUT FOR MATCHING.
3800 C  NQ PRE-DETERMINED MODEL COEFFS FROM DATA USED IN FIT OF POLY.
3900 C  NR COEFFS. INCLUDED BY REGRESSION. POLY. DEGREE P=NQ+NR.
4000 C-----------------------------------------------------------------
4100          DATA NB,NQ,NR/1,0,3/
4200 C ** CONVERSION CONSTANT -C FROM PERMEABILITY TO UNITS OF CONDUCTIVITY
4300 C ** (CM/DAY).   MILLINGTON-QUIRK POWER -P.
4400          C=1.88E04*60.*24. ; P=4./3.
4500 C ** OPTION TO COMPUTE CONDUCTIVITY AS FUNCTION OF SATURATION:
4600 C ** SET IASK=1.
4700          IASK=1; THSATO=THSAT
4800          IF(IASK.NE.1) GO TO 1; THFULL=THFULL/THSAT;THLOW=THLOW/THSAT
```

```
4900          THO=THO/THSAT;THKMIN=THKMIN/THSAT;THOUT=THOUT/THSAT
5000          B(1)=B(1)*THSAT; RHO=RHO*THSAT; C=C*THSAT**P; THSAT=1.0
5100 C ** COMPUTE THE SQUARE RADIUS OF EACH PORE CLASS.
5200    1     DELTH=THSAT/N; NFULL=N
5300          DO 2 I=1,N; TH=(N-I)*DELTH+DELTH; THMID=TH-DELTH/2.
5400          IF(THMID.GT.THFULL)GO TO 20
5500          IF(THMID.LT.THLOW)GO TO 30; M=M+1
5600          THETA(M)=TH; HD=HFN(THMID,THSAT,A,RHO)
5700          RSQ(M)=1./HD**2; H(M)=HFN(TH,THSAT,A,RHO)
5800          GO TO 2
5900    20    NFULL=NFULL-1
6000    2     CONTINUE
6100    30    N=NFULL; C=C/N**P
6200 C ** CALCULATE MILLINGTON-QUIRK CONDUCTIVITY FOR NFULL PORE
6300 C ** CLASSES BELOW FULL VALUE OF THETA.
6400 C ** SUM TERMINATES FOR M CLASSES WITH THEIR ABOVE THLOW.
6500          DO 4 I=1,M; SUM=0; DO 3 J=I,M
6600    3     SUM=SUM+(2*J+1-2*I)*RSQ(J)
6700          WK(I)=C*(THETA(I)**P)*SUM
6800    4     CONTINUE
6900 C ** CALCULATE BEST FIT MATCHING FACTOR AT THETA OF PORE CLASSES.
7000          NP=0; SUMY=0; SUMYY=0
7100          DO 5 I=1,M; TH=THETA(I)
7200          IF(TH.GT.THO.OR.TH.LT.THKMIN) GO TO 5
7300          NP=NP+1; CON=CFN(TH,THO,COND,B,NB)
7400          Y=ALOG(CON/WK(I))
7500          SUMY=SUMY+Y; SUMYY=SUMYY+Y*Y
7600    5     CONTINUE
7700          F=SUMY/NP; VAR=SUMYY-NP*F**2
7800          IF(NP.GT.1) VAR=VAR/(NP-1); STD=SQRT(VAR)
8000          PRINT/,'************************************************'
8100          PRINT/,'  MILLINGTON-QUIRK CONDUCTIVITY'
8200          PRINT/,'************************************************'
8300          PRINT/,' '
8400          PRINT/,' MATCHING FACTOR F= K ACT./K CALC.'
8500          PRINT/,' MEAN LOG F=',F,' STD. DEV. LOG F=',STD
8600          F=EXP(F); FGEO=F
8700          PRINT/,' GEOMETRIC MEAN F=',F
8750 C ** ASSUME  A LOG-NORMAL DISTRIBUTION OF MATCHING FACTORS.
8800          E=EXP(VAR/2.); F=F*E; STD=F*SQRT(E**2-1.)
8900          PRINT/,' EST. MEAN F=',F,' EST. STD. DEV. F=',STD
9000          PRINT/,' NUMBER OF POINTS MATCHED =',NP,' D.F.=',NP-1
9100          PRINT/,' '
9200          PRINT/,' CONDUCTIVITY FOR NUMBER OF PORE CLASSES =',N
9300          IF(IASK.EQ.1)PRINT/,' COMPUTED AS FUNCTION OF SATURATION'
9400          PRINT/,' '
9500          PRINT &
9600    6     FORMAT(1X,'WATER CONT',3X,'SAT.',2X,'HEAD CM',2X,'CONDUCTIVITY')
9700          DO 7 I=1,M; WK(I)=FGEO*WK(I); SAT=THETA(I)/THSAT
9800          IF(THETA(I).LT.THOUT) GO TO 10
9900          IF(IASK.EQ.1)THETA(I)=THETA(I)*THSATO
10000   7      PRINT 8, THETA(I),SAT,H(I),WK(I)
10100   8     FORMAT(1X,6X,F4.3,2X,F5.3,3X,F6.0,4X,E10.4)
10200   10    M=I-1
10300         PRINT/,' '; PRINT/,' THETA LIMITS:'
10400         PRINT/,' THSAT=',THSATO,' THFULL=',THFULL,' THO=',THO
10500         PRINT/,' THKMIN=',THKMIN,' THLOW=',THLOW,' THOUT=',THOUT
10600         PRINT/,' MODEL PARAMETERS:'
10700         PRINT/,' HEAD ','A=',A,'RHO=',RHO
10800         PRINT/,' CONDUCTIVITY ','KO=',COND,'BETA=',B(1)
10900         WRITE(1,/) M,THSATO, (THETA(I),I=1,M)
11000         WRITE(1,/) (H(I),I=1,M)
11100         WRITE(1,/) (WK(I),I=1,M); LOCK 1
11200         IF(NR.EQ.0) GO TO 100
11300 C ** FIT POLYNOMIAL  TO LOG K CALCULATED BY M-Q METHOD.
11400         ALOGKO=ALOG(COND); DO 35 I=1,M
11500         IF(IASK.EQ.1) THETA(I)=THETA(I)/THSATO
11600         U(I)=THETA(I)-THO; V(I)=ALOG(WK(I))-ALOGKO
11700         IF(NQ.EQ.0) GO TO 35; TH=1.
11800         DO 35 J=1,NQ; TH=TH*U(I); V(I)=V(I)-B(J)*TH
11900   35    CONTINUE
11950         PRINT/,' '; PRINT/,' '
12000         PRINT/,' REGRESSION FOR MILLINGTON-QUIRK  K'
12100         PRINT/,' LOG K=LOG KO +B(1)(THETA-THO)+...+B(P)(THETA-THO)**P'
12200         PRINT/,' DEGREE P=',NQ+NR
12300         PRINT/,' '
12400         PRINT/,' PRE-DETERMINED COEFFICIENTS:'
12500         PRINT/,' KO=',CONO #IF(NQ.EQ.0) GO TO 50
12550         DO 45 J=1,NQ
12600   45    PRINT 40, J, B(J)
```

```
12700    40    FORMAT(2X,'B(',I2,')=',E12.6)
12800          PRINT/,' '
12900    50    NQ=NQ+1
13000          PRINT/,' COEFFS. INCLUDED BY REGRESSION FOR M-Q K'
13100          DO 60 J=1,NR
13200          PRINT/,'************************************************'
13300          CALL PREGS(U,V,M,BETA,NQ,J)
13400    60    PRINT/,'************************************************'
13500          NQ=NQ-1; DO 80 J=1,NR; K=NQ+J
13510    80    B(K)=BETA(J)
13600          PRINT 65
13700    65    FORMAT(1X,7X,'THETA',7X,'M-Q K',2X,'REG. CALC. K')
13800          NR=NQ+NR; DO 70 I=1,M
13900          COND=CFN(THETA(I),THO,CONO,B,NR)
14000    70    PRINT 75, THETA(I),WK(I),COND
14100    75    FORMAT(1X,7X,F5.3,2(2X,E10.4))
14200   100    STOP; END
14300 C ** MODEL FUNCTIONS DEFINED BELOW
14400          FUNCTION HFN(THETA,THSAT,A,RHO)
14500          HFN=A*(EXP(RHO*(THETA-THSAT))-1.)
14600          IF(HFN.LT.1.) HFN=1.
14700          RETURN
14800          END
14900          FUNCTION CFN(THETA,THO,CONO,B,N)
15000          DIMENSION B(N)
15100          TH=THETA-THO; THETAJ=1.; S=0.
15200          DO 1 J=1,N; THETAJ=TH*THETAJ
15300    1     S=S+B(J)*THETAJ; CFN=CONO*EXP(S)
15400          RETURN
15500          END
#
L
100 $RESET FREE
200 $SET SEPARATE
300 $SET LIBRARY
400          SUBROUTINE PREGS(X,Y,N,B,NQ,NP)
500          DIMENSION X(N),Y(N),B(NP)
600          DIMENSION Z(50,10),C(10,10), D(10)
700 C ** POLYNOMIAL REGRESSION: Y=(B(1)+B(2)X+...+B(NP)X**(NP-1)) X**NQ
900          DO 1 I=1,N; XMULT=X(I)**NQ; DO 1 J=1,NP; Z(I,J)=XMULT
1000    1     XMULT=XMULT*X(I)
1100 C ** COMPUTE CROSS PRODUCT MATRIX.
1200          DO 2 I=1,N; DO 2 J=1,NP; D(J)=D(H)+Z(I,J)*Y(I)
1300          DO 2 K=1,NP; C(J,K)=C(J,K)+Z(I,J)*Z(I,K)
1400    2     CONTINUE
1500 C ** SOLVE FOR REGRESSION COEFFS. B.
1600          CALL RSOLVE(C,D,B,NP)
1900          DO 5 I=1,N; YR=0; SUMY=SUMY+Y(I)
2000          SUMYY=SUMYY+Y(I)**2; DO 4 J=1,NP
2100    4     YR=YR+B(J)*Z(I,J); SSE=SSE+(YR-Y(I))**2
2200    5     CONTINUE
2300          SST=SUMYY-SUMY**2/N; SSR=SST-SSE
2400          SE=SQRT(SSE/(N-NP)); R=SQRT(SSR/SST)
2500          WRITE(6,6)
2600    6     FORMAT(1X,'REGRESSION COEFFICIENTS:',/)
2700          DO 7 J=1,NP
2800    7     WRITE(6,8) NQ+J-1, B(J)
2900    8     FORMAT(1X,'B(',I2,')=',E12.6)
3000          PRINT/,' '
3100          PRINT/,' STD. ERROR OF ESTIMATE =',SE,'D.F.=',N-NP
3200          PRINT/,' MULT. CORRELATION COEFFICIENT R=',R
3300          RETURN
3400          END
4800          SUBROUTINE RSOLVE(A,Y,X,N)
4900 C        SOLVES THE MATRIX EQUATION A*X=Y FOR X
5000 C        A IS AN N BY N MATRIX.
5100 C        ORIGINAL MATRIX A IS DESTROYED.
5200          DIMENSION A(10,11),Y(N),X(N),LOC(10),CK(10)
5300          NP=N+1
5400          DO 1 I=1,N; A(I,NP)=Y(I)
5500    1     CK(I)=0.0
5550          DO 100 I=1,N
5600          IP=I+1
5700 C        FIND MAX ELEMENT IN I-TH COL
5800          AMAX=0.0; DO 2 K=1,N; IF(AMAX-ABS(A(K,I)))3,2,2
5900 C        IS NEW MAX IN ROW PREVIOUSLY USED AS PIVOT
6000    3     IF(CK(K))4,4,2
6100    4     LOC(I)=K; AMAX=ABS(A(K,I))
6200    2     CONTINUE
6300 C        MAX ELEMENT IN I-TH COL IS A(L,I)
```

```
6500    5     L=LOC(I); CK(L)=1.
6600 C        PERFORM ELIMINATION, L IS PIVOT ROW, A(L,I) IS PIVOT ELEMENT
6700          DO 50 J=1,N; IF(L-J)6,50,6
6800    6     F=-A(J,I)/A(L,I)
6900          DO 40 K=IP,NP
7000    40    A(J,K)=A(J,K)+F*A(L,K)
7100    50    CONTINUE
7200    100   CONTINUE
7300          DO 200 I=1,N; L=LOC(I)
7400    200   X(I)=A(L,NP)/A(L,I)
7500          RETURN
7600          END
$
```

APPENDIX F

Correction methods for scaling with an approximate saturation variable.

The effect of using an approximate saturation variable s equal to θ/θ_o for estimation of scale factors will be considered. Here the field measured saturated water content θ_o (initial value) is less than the actual saturation value ϕ.

Conductivity:

A corrected scale mean conductivity function in terms of the actual saturation s ' equal to θ/ϕ for each location is

$$K'(s') = K'_m e^{b'(s'-s_o)} \tag{1}$$

where

$$K'^{1/2}_m = \frac{1}{R} \Sigma K'^{1/2}_o \tag{2}$$

$$K'_o = K_o \exp[b(s_o\phi/\theta_o - 1)] \tag{3}$$

and

$$b' = (\frac{1}{R} \Sigma \phi/\theta_o)b = b/s_o \tag{4}$$

for the sums taken over R locations. K_o and b are the original parameters of the model. New scales ω' satisfy

$$K'_o = \omega'^2 K'_m \tag{5}$$

for each location with average equal 1. Let $b_o = b\phi/\theta_o$ for each location and

$$\sigma_b^2 = \frac{1}{R-1} \Sigma (b_o - b')^2 \tag{6}$$

where the sum is over R locations. Then σ_b represents the error in replacing b by a new common parameter b', where the original conductivity in terms of

approximate saturation s is

$$K(s) = K_o\, e^{b(s-1)} = K'_o\, e^{b_o(s'-s_o)}. \tag{7}$$

Thus (7) can be viewed as an approximation to

$$K'(s') = K'_o\, e^{b'(s'-s_o)} \tag{8}$$

which is scaled relative to s'. The difference in (7) and (8) satisfies

$$\ln K'(s') - \ln K(s) = (b' - b_o)(s' - s_o) \tag{9}$$

and the error over locations made by replacing (7) by (8) is

$$\left[\frac{1}{R-1}\sum_{r=1}^{R} (\ln K'_r - \ln K_r)^2\right]^{1/2} = \sigma_b\, |s' - s_o|. \tag{10}$$

Next consider the situation for which the scale factors are not altered. Suppose that

$$\theta_o/\phi = s_o$$

for all locations, i.e., the ratio is constant. Then K'_o equals K_o for all locations, and the scales ω' are identical to those obtained relative to the approximate saturation variable.

Pressure head:

Suppose that the pressure head is scaled in terms of the function

$$h(s) = a(e^{b(s-1)}-1) \tag{11}$$

where s is an approximate saturation. Define

$$C(s) = \frac{dh(s)}{ds}. \tag{12}$$

The quantity (12) can be scaled if the pressure head (11) can, that is, $\alpha h(s) = h_m(s)$ implies $\alpha C(s) = C_m(s)$.

An approximate scaling of pressure head relative to actual saturation s' is obtained as follows:

$$C(s) = \left(a\,\frac{\theta_o}{\phi}\, e^{b(\phi/\theta_o-1)}\right) b_o\, e^{b_o(s'-1)} \tag{13}$$

where $b_o = b\phi/\theta_o$ for each location. Let

$$b' = \frac{1}{R} \Sigma\, b_o = \frac{b}{R} \Sigma\, \phi/\theta_o \tag{14}$$

where the sum is over R locations. Now define

$$a' = a \frac{\theta_o}{\phi} e^{b(\phi/\theta_o - 1)} . \tag{15}$$

Then

$$C'(s') = a'\, b'\, e^{b'(s'-1)} \tag{16}$$

is an approximation to (13). The error made in replacing b by b' equals the square root of the variance of b_o over locations. Integration of (16) yields the corrected pressure head:

$$h'(s') = a'\, (e^{b'(s'-1)} - 1) \tag{17}$$

where a' depends on location and b' is common to all locations. Rescaling error equals $|h'(s') - h(s)|$. Since the new scale factors depend only on (15), they are identical to the original scales if θ_o/ϕ is constant over locations.

ACKNOWLEDGMENTS

The unstinted interest and effort of students and technical staff personnel of the Department of Land, Air and Water Resources are gratefully acknowledged. This work was partially supported by the Kearney Foundation of Soil Science and the National Science Foundation.

3m9,'79 (PAD/SL)